The Scalar Product

Introduction

The Scalar(or Dot Product), of two vectors **a** and **b** is written

$$\mathbf{a}.\mathbf{b}$$

If the two vectors are inclined to each other by an angle(say θ) then the product is written

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|.|\mathbf{b}|\cos\theta \quad \text{or} \quad \mathbf{a}.\mathbf{b} = ab\cos\theta$$

Even though the left hand side of the equation is written in terms of vectors, the answer is a scalar quantity.

Rules

$\mathbf{a}.\mathbf{b} = ab\cos\theta = \mathbf{b}.\mathbf{a}$

when **a** & **b** are parallel, $\theta = 0$, $\cos\theta = 1$, $\mathbf{a}.\mathbf{b} = ab$
(unit vectors $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$)

when **a** & **b** are at 90°, $\theta = 90°$, $\cos\theta = 0$, $\mathbf{a}.\mathbf{b} = 0$
(unit vectors: $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$ $\mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$ $\mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = 0$)

if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\mathbf{a}|^2 = \mathbf{a}.\mathbf{a} = a_1^2 + a_2^2 + a_3^2$$

$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$ $\mathbf{a}.(\mathbf{b} - \mathbf{c}) = \mathbf{a}.\mathbf{b} - \mathbf{a}.\mathbf{c}$

$(\mathbf{a} + \mathbf{b}).\mathbf{c} = \mathbf{a}.\mathbf{c} + \mathbf{b}.\mathbf{c}$ $(\mathbf{a} - \mathbf{b}).\mathbf{c} = \mathbf{a}.\mathbf{c} - \mathbf{b}.\mathbf{c}$

$(\lambda\mathbf{a}).\mathbf{b} = \lambda(\mathbf{a}.\mathbf{b}) = \mathbf{a}.(\lambda\mathbf{b})$

$$\cos\theta = \frac{a.b}{|a|.|b|}$$

Example #1

Given that **a** = 3**i** - **j** + 2**k** and **b** = 2**i** + **j** - 2**k** ,
find **a**.**b** and the included angle between the vectors to 1 d.p.

$$\begin{aligned} a.b &= (3i - j + 2k).(2i + j - 2k) \\ &= 6 - 1 - 4 \\ &= 1 \end{aligned}$$

$\underline{a.b = 1}$

$$\begin{aligned} |a^2| &= a.a \\ &= (3i - j + 2k).(3i - j + 2k) \\ &= 9 + 1 + 4 \\ &= 14 \\ |a| &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} |b^2| &= b.b \\ &= (2i + j - 2k).(2i + j - 2k) \\ &= 4 + 1 + 4 \\ &= 9 \\ |b| &= 3 \end{aligned}$$

with θ the included angle

$$\cos\theta = \frac{a.b}{|a|.|b|}$$

substituting for $a.b$, $|a|$, $|b|$

$$\cos\theta = \frac{1}{\sqrt{14} \times 3} = \frac{1}{11.225}$$

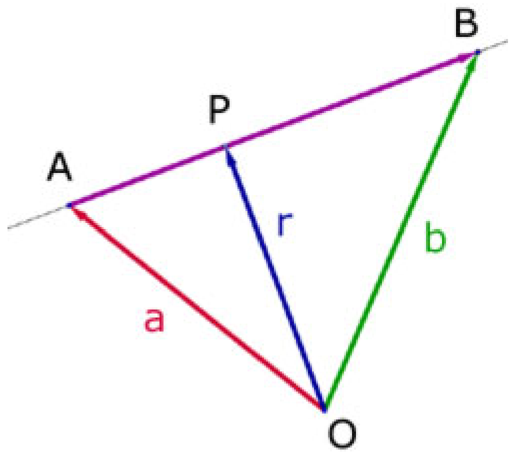
$$\Rightarrow \quad \theta = 84.89^\circ$$

<u>the angle between a and b is 84.9° (1 d.p.)</u>

Example #2

What is the vector equation describing the straight line passing through the points A(-8, 1, -2) and B(10, -1, 3)?

Find the coordinates of a point P on AB such that OP is perpendicular to AB(origin O), hence find the distance OP to 2d.p.



given points: A(-8, 1, -2) B(10, -1, 3)

$\Rightarrow \overrightarrow{AB} = (-8+10)i + (1-1)j + (-2+3)k$

$= 2i + 0 + k$

$= 2i + k$

if r is the position vector for point P

the vector equation of line AB is:

$r = a + \lambda(\overrightarrow{AB})$

$r = a + \lambda(2i + k)$

$= -8i + j - 2k + \lambda(2i + k)$

$= -8i + j - 2k + 2\lambda i + \lambda k$

$= (2\lambda - 8)i + j + (\lambda - 2)k$

$\therefore$ coords. of any point P on AB are $(2\lambda - 8, 1, \lambda - 2)$

for $\overrightarrow{OP}$ to be perpendicular to $\overrightarrow{AB}$

$\Rightarrow \overrightarrow{OP} \,.\, \overrightarrow{AB} = 0$

$\Rightarrow [(2\lambda - 8)i + j + (\lambda - 2)k] \,.\, [(2i + k)] = 0$

$\Rightarrow 2(2\lambda - 8) + 0 + 1(\lambda - 2) = 0$

$\Rightarrow 4\lambda - 16 + \lambda - 2 = 0$

$\Rightarrow 5\lambda = 18$

$\Rightarrow \lambda = 3.6$

substituting into coords. for $P(2\lambda - 8, 1, \lambda - 2)$

$\Rightarrow P(2 \times 3.6 - 8,\ 1,\ 3.6 - 2)$

$\Rightarrow P(-0.8,\ 1,\ 1.6)$

$\therefore |\overrightarrow{OP}| = \sqrt{(-0.8)^2 + (1)^2 + (1.6)^2}$

$= \sqrt{(0.64) + (1) + (2.56)}$

$= 2.04939$

perp. dist. between O and AB is 2.05 (2d.p.)