Vectors : Vector Equations

Component rules

Consider two vectors:

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad \text{and} \quad \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

in three dimensional space.

a = **b** implies that $x_1 = x_2$ $y_1 = y_2$ $z_1 = z_2$

$\mathbf{a} + \mathbf{b} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}$

$\mathbf{a} - \mathbf{b} = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k}$

$m\mathbf{a} = mx_1\mathbf{i} + my_1\mathbf{j} + mz_1\mathbf{k}$ where m is a scalar quantity

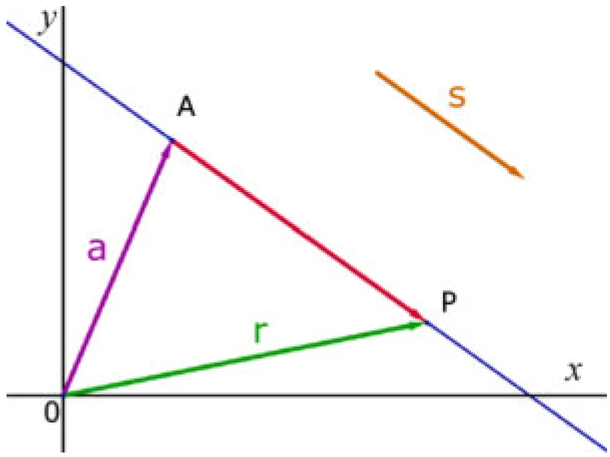
$|\mathrm{a}| = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$ $|\mathrm{b}| = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$

in 3D space, if point A has position vector **a** and point B has position vector **b** then the distance AB is given by:

$$\mathrm{AB} = |\overrightarrow{\mathrm{AB}}| = \mathrm{b} - \mathrm{a}$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Equation straight line - single point & parallel vector given



A(x_1 y_1 z_1) is a fixed point on the line

a is the position vector for point A **a** = x_1**i** + y_1 **j** + z_1**k**

s is a vector parallel to the line **s** = *l* **i** + *m* **j** + *n* **k**
(*l m n* are called the direction ratios of the line)

r is the position vector for an arbitrary point P(x, y, z) on the line.

$\overrightarrow{AP}$ is parallel to s

$\Rightarrow \overrightarrow{AP} = \sigma s$ (σ is a scalar variable)

$\Rightarrow \underline{r = a + \sigma s}$ (vector equation of the line)

substituting for a and s

$r = x_1 i + y_1 j + z_1 k + \sigma(li + mj + nk)$

explanding and rearranging

$r = (x_1 + \sigma l)i + (y_1 + \sigma m)j + (z_1 + \sigma n)k$

$r = xi + yj + zk$

comparing the coefficients of i j k

$x = x_1 + \sigma l \qquad y = y_1 + \sigma m \qquad z = z_1 + \sigma n$

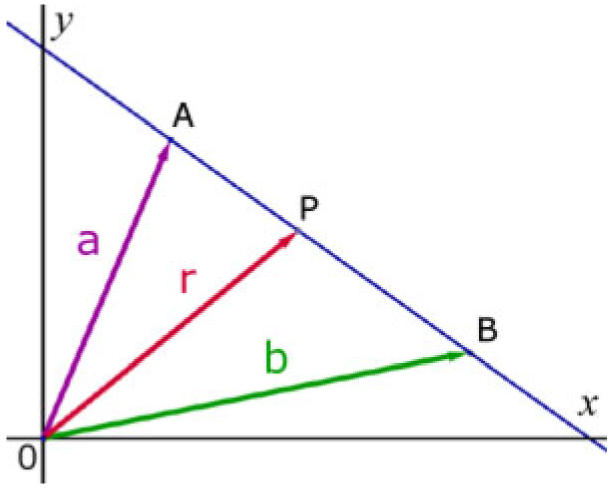
rearranging to make σ the subject

$$\sigma = \frac{x - x_1}{l}, \qquad \sigma = \frac{y - y_1}{m}, \qquad \sigma = \frac{z - z_1}{n}$$

hence the Cartesian equation of the line:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (= \sigma)$$

Equation of a straight line - two points given



A(x_1 y_1 z_1) is a fixed point on the line

a is the position vector for point A **a** = x_1**i** + y_1 **j** + z_1**k**

B(x_2 y_2 z_2) is a fixed point on the line

b is the position vector for point B **b** = x_2**i** + y_2 **j** + z_2**k**

r is the position vector for an arbitrary point P(x, y, z) on the line.

let $\overrightarrow{AB}$ be a vector from A to B

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

since $\quad b = \overrightarrow{OB} \quad$ and $\quad a = \overrightarrow{OA}$

$\therefore \quad \overrightarrow{AB} = b - a \qquad\qquad (i$

$\overrightarrow{AP} = \theta\overrightarrow{AB} \qquad\qquad$ (θ is a scalar variable)

substituting for $\overrightarrow{AB}$ from (i

$\overrightarrow{AP} = \theta(b - a)$

$\overrightarrow{OP} = r$

$r = \overrightarrow{OA} + \overrightarrow{AP}$

$\underline{r = a + \theta(b - a)} \qquad$ (vector equation of the line)

replacing the vectors by their components

$$xi + yj + zk = x_1i + y_1j + z_1k + \theta(x_2i + y_2j + z_2k$$
$$- x_1i + y_1j + z_1k)$$

expanding

$$xi + yj + zk = x_1i + y_1j + z_1k + \theta x_2i + \theta y_2j + \theta z_2k$$
$$- \theta x_1i + \theta y_1j + \theta z_1k$$

factorising

$$xi + yj + zk = [x_1 + \theta(x_2 - x_1)]i + [y_1 + \theta(y_2 - y_1)]j$$
$$+ [z_1 + \theta(z_2 - z_1)]k$$

comparing the coefficients of i j k

$x = x_1 + \theta(x_2 - x_1)$

$y = y_1 + \theta(y_2 - y_1)$

$z = z_1 + \theta(z_2 - z_1)$

making θ the subject

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (= \theta)$$

(the Cartesian equations)