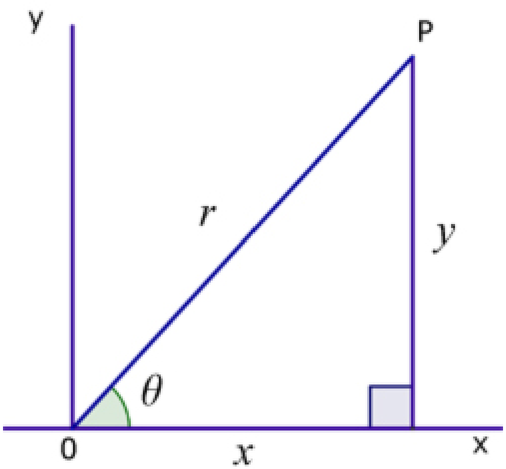
Pythagorean Identities

identity #1 $\cos^2\theta + \sin^2\theta \equiv 1$



by Pythagoras' Theorem:

$$x^2 + y^2 = r^2 \qquad \text{(i}$$

dividing (i by r^2

$$\Rightarrow \qquad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

but $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

$$\therefore \qquad \cos^2\theta + \sin^2\theta \equiv 1$$

Example

solve the equation $2\sin^2\theta = 1 - \cos\theta$

for θ where $0 \le \theta < 360°$

$2\sin^2\theta = 1 - \cos\theta$ (i

using $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = 1 - \cos^2\theta$

substituting for $\sin^2\theta$ into (i

$$2(1-\cos^2\theta) = 1 - \cos\theta$$

$$2 - 2\cos^2\theta = 1 - \cos\theta$$

$$0 = 1 - \cos\theta - 2 + 2\cos^2\theta$$

$$0 = 2\cos^2\theta - \cos\theta - 1$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$\therefore \quad \cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\Rightarrow \quad \underline{\theta = 60°,\ 120°,\ 240° \quad \text{or} \quad \theta = 0°}$

identity #2 $1+\tan^2\theta \equiv \sec^2\theta$

dividing $\cos^2\theta + \sin^2\theta \equiv 1$ by $\cos^2\theta$

$$1+\frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

but $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

$\therefore$ $\underline{1+\tan^2\theta \equiv \sec^2\theta}$

Example

solve the equation $3\tan\theta = \sec^2\theta + 1$

for $0 \le \theta \le 360°$

rearranging

$$0 = \sec^2\theta - 3\tan\theta + 1$$

$$\sec^2\theta - 3\tan\theta + 1 = 0 \qquad (i$$

using the identity

$$1+\tan^2\theta \equiv \sec^2\theta$$

substituting for $\sec^2\theta$ into (i

$$(1+\tan^2\theta) - 3\tan\theta + 1 = 0$$

$$\tan^2\theta - 3\tan\theta + 2 = 0$$

$$(\tan\theta - 1)(\tan\theta - 2) = 0$$

$\tan\theta = 1$ or $\tan\theta = 2$

$\therefore$ $\theta = 45°, 225°$ or $\theta = 63.4°, 243.4$

identity #3 $\cot^2 + 1 \equiv \text{cosec}^2\,\theta$

dividing $\cos^2\theta + \sin^2\theta \equiv 1$ by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

but $\text{cotan}\,\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

$$\therefore \qquad \cot^2\theta + 1 \equiv \text{cosec}^2\theta$$

Example

simplify $\left[x^2\left(\text{cosec}^2\theta - 1\right)\right]^{-1}$

expanding

$$\left[x^2\left(\text{cosec}^2\theta - 1\right)\right]^{-1} \equiv \frac{1}{x^2\left(\text{cosec}^2\theta - 1\right)}$$

$$\equiv \frac{1}{x^2\text{cosec}^2\theta - x^2} \qquad \text{(i}$$

using the identity

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

and substituting for $\text{cosec}^2\theta$ into (i

$$\left[x^2\left(\text{cosec}^2\theta - 1\right)\right]^{-1} \equiv \frac{1}{x^2(\cot^2\theta + 1) - x^2}$$

$$\equiv \frac{1}{x^2\cot^2\theta + x^2 - x^2}$$

$$\equiv \frac{1}{x^2\cot^2\theta}$$

but $\tan\theta = \dfrac{1}{\cot\theta}$

$$\therefore \qquad \left[x^2\left(\text{cosec}^2\theta - 1\right)\right]^{-1} \equiv \frac{\tan^2\theta}{x^2}$$