The Coefficient of Restitution

The **Coefficient of Restitution (e)** is a variable number with no units, with limits from zero to one.

$$0 \le e \le 1$$

'e' is a consequence of **Newton's Experimental Law of Impact**, which describes how the speed of separation of two impacting bodies compares with their speed of approach.

note: the speeds are **relative** speeds

$$v_a = \text{speed of approach} \qquad v_s = \text{speed of separation}$$

$$e = \text{coefficient of restitution}$$

$$\text{coefficient of restitution} = \frac{\text{speed of separation}}{\text{speed of approach}}$$

$$e = \frac{v_s}{v_a}$$

If we consider the speed of individual masses before and after collision, we obtain another useful equation:

u_A = initial speed of mass A

u_B = initial speed of mass B

v_A = final speed of mass A

v_B = final speed of mass B

relative initial speed of mass A to mass B = $u_B - u_A$

relative final speed of mass A to mass B = $v_B - v_A$

$$\text{coefficient of restitution (e)} = \frac{v_B - v_A}{u_B - u_A}$$

note: in this equation the absolute of $u_B - u_A$ and $v_B - v_A$ are used (|absolute| no net negative result)

Example

A 5 kg mass moving at 6 ms^{-1} makes a head-on collision with a 4 kg mass travelling at 3 ms^{-1} .
Assuming that there are no external forces acting on the system, what are the velocities of the two masses after impact?
(assume coefficient of restitution e = 0.5)

u_A = initial speed of 5 kg mass (mass A)= 6 ms^{-1}

u_B = initial speed of 4 kg mass (mass B)= 3 ms^{-1}

m_A = 5 kg m_B = 4 kg

v_A = final speed of mass A v_B = final speed of mass B

momentum before the collision equals momentum after

hence, $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

also

$$\text{coefficient of restitution (e)} = \frac{v_B - v_A}{u_B - u_A}$$

substituting for e, m_A , u_A , m_B , u_B

we obtain two simultaneous equations

from the conservation of momentum,

$$5 \times 0.6 + 4(-3) = 5\,v_A + 4\,v_B$$

$$3 - 12 = 5\,v_A + 4\,v_B$$

$$-9 = 5\,v_A + 4\,v_B$$

$$5\,v_A + 4\,v_B = -9 \qquad \text{(i}$$

from the coefficient of restitution expression,

$$0.5 = \frac{v_B - v_A}{u_B - u_A}$$

$$0.5(u_A - u_B) = v_B - v_A$$

$$(0.5 \times 6) - (0.5(-3)) = v_B - v_A$$

$$3 + 1.5 = v_B - v_A$$

$$4.5 = v_B - v_A$$

$$v_B - v_A = 4.5 \qquad \text{(ii}$$

multiplying (ii by 5 and adding

$$5\ v_A + 4\ v_B = -9$$

$$-\ 5\ v_A + 5\ v_B = 22.5$$

$$9\ v_B = 13.5$$

$$\mathbf{v_B = 1.5\ ms^{-1}}$$

from (ii

$$1.5 - v_A = 4.5$$

$$v_A = 1.5 - 4.5 = -3$$

$$\mathbf{v_A = -3\ ms^{-1}}$$

Ans. The velocities of the 5 kg and 4 kg masses are -3 ms^{-1} and 1.5 ms^{-1}, respectively.

Oblique Collisions

For two masses colliding along a line, Newton's Experimental law is true for component speeds. That is, the law is applied twice: to each pair of component speeds acting in a particular direction.

Example

A particle of mass *m* impacts a smooth wall at 4*u* ms^{-1} at an angle of 30 deg. to the vertical. The particle rebounds with a speed *ku* at 90 deg. to the original direction and in the same plane as the impact trajectory.

What is:
i) the value of the constant '*k*' ?
ii) the coefficient of restitution between the wall and the particle?
iii) the magnitude of the impulse of the wall on the particle

i) There is no momentum change parallel to the wall.

$$m(4u)\cos 30^\circ = m(ku)\cos 60^\circ$$

$$4u\frac{\sqrt{3}}{2} = ku.\frac{1}{2}$$

$$4\sqrt{3} = k, \qquad \underline{k = 4\sqrt{3}}$$

ii) The coefficient of restitution'e' is the ratio of the speed of separation to the speed of approach:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{ku\sin 60^\circ}{4u\sin 30^\circ}$$

$$= \frac{4\sqrt{3}.\frac{\sqrt{3}}{2}u}{2u} = \frac{6u}{2u} = 3$$

$$\underline{e = 3}$$

iii) The impulse is the change of momentum.

Since the vertical unit vectors are unchanged, the momentum change just concerns the horizontal vector components.

hence,

$$\begin{aligned}\text{momentum change } &= m(4u\sin 30^\circ) - m(-ku\sin 60^\circ)\\ &= m.4u.\frac{1}{2} + m.4\sqrt{3}u\,\frac{\sqrt{3}}{2}\\ &= 2mu + 6mu\\ &= 8mu\end{aligned}$$

<u>ans. momentum change is $8mu$ N.s</u>