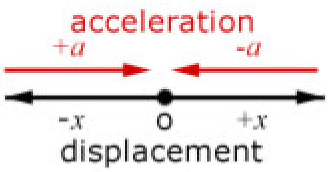
Linear Motion : Simple Harmonic Motion

Theory

A particle is said to move with S.H.M when the acceleration of the particle about a fixed point is proportional to its displacement but opposite in direction.



Hence, when the displacement is positive the acceleration is negative(and vice versa).

This can be described by the equation:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where x is the displacement about a fixed point O(positive to the right, negative to the left), and w^2 is a positive constant.

An equation for velocity is obtained using the expression for acceleration in terms of velocity and rate of change of velocity with respect to displacement(see 'non-uniform acceleration').

$$v\frac{dv}{dx} = -\omega^2 x$$

separating the variable and integrating,

$$\int v\,dv = \int -\omega^2 dx$$
$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2 x^2 + C$$

$v = 0 \quad \text{when} \quad x = a$

$\Rightarrow \quad \underline{v^2 = \omega^2(a^2 - x^2)}$

$\Rightarrow \quad v = \pm\omega(a^2 - x^2)^{\frac{1}{2}}$

but $\quad v = \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dx}{dt} = \pm\omega(a^2 - x^2)^{\frac{1}{2}}$

separating the variable and integrating again,

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \int \pm\omega\,dt$$
$$-\cos^{-1}\left(\frac{x}{a}\right) = \pm\omega t + C$$

when $\; t = 0 \;$ when $\; x = a$

$\Rightarrow \quad C = 0$

$\therefore \quad \cos(\omega t) = \dfrac{x}{a} \quad \text{or} \quad \underline{x = a\cos(\omega t)}$

NB $\cos^{-1}()$ is the same as arc cos()

So the displacement against time is a cosine curve. This means that at the end of one completete cycle,

$$\omega T = 2\pi \qquad (T \text{ is the period})$$
$$\therefore \quad \underline{T = \frac{2\pi}{\omega}}$$

Example

A particle displaying SHM moves in a straight line between extreme positions A & B and passes through a mid-position O.

If the distance AB=10 m and the max. speed of the particle is 15 m^{-1} find the period of the motion to 1 decimal place.

$\text{AB} = 10\text{m} \quad \therefore \quad \text{amplitude} \;\; a = 5\text{m}$

$\text{max. speed} \qquad v_{max} = 15\text{ms}^{-1}$

$$v^2 = \omega^2(a^2 - x^2)$$

when v is max., displacement $x = 0$

$\therefore \quad v^2 = \omega^2 a^2$

$\Rightarrow \quad v = \omega a$

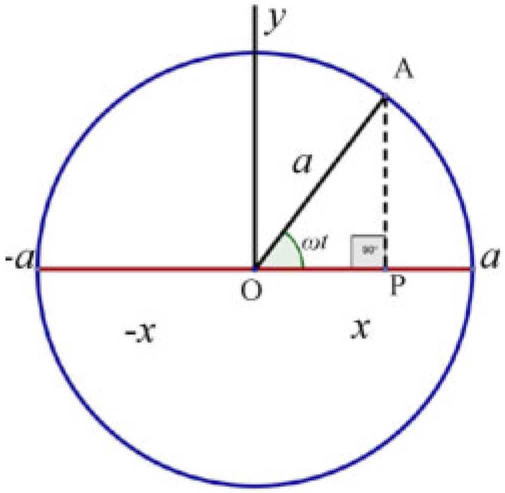
$\Rightarrow \quad \omega = \dfrac{v}{a} = \dfrac{15}{5} = 3$

period T is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3} = 2.094$$

Ans. period of motion is 2.1 secs.

SHM and Circular Motion



The SHM-circle connection is used to solve problems concerning the time interval between particle positions.

To prove how SHM is derived from circular motion we must first draw a circle of radius 'a'(max. displacement).

Then, the projection(x-coord.) of a particle A is made on the diameter along the x-axis. This projection, as the particle moves around the circle, is the SHM displacement about O.

from triangle OAP

$$x = a\cos\omega t \qquad (i$$

$$\frac{dx}{dt} = -a\omega\sin\omega t$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos\omega t$$

but $a\cos\omega t = x$ from (i

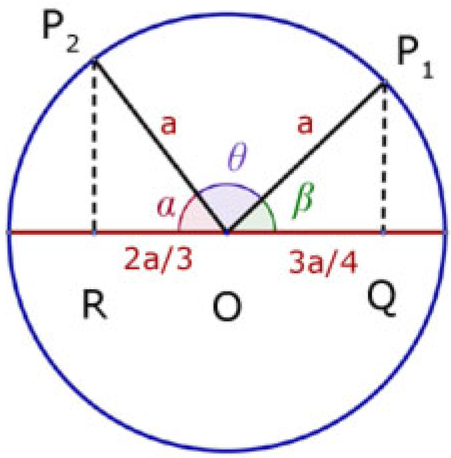
$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\omega^2 x$$

Q.E.D. the motion of particle P along the x-axis is S.H.M.

Example

A particle P moving with SHM about a centre O, has period T and amplitude a .

Q is a point $3a/4$ from O. R is a point $2a/3$ from O.What is the time interval(in terms of T) for P to move directly from Q to R? Answer to 2 d.p.



let the time interval between P_1 and P_2 be t secs.
let angle P_1OP_2 be θ rads.

period T is given by: $T = \frac{2\pi}{\omega}$, $\omega = \frac{2\pi}{T}$ (i

$\Rightarrow$ $t = \frac{\theta}{\omega}$ (ii

the angles in a straight line = 180 deg.
$\therefore$ from the diagram, $\pi = \theta + \alpha + \beta$
$\Rightarrow$ $\theta = \pi - \alpha - \beta$

from (ii substituting for θ

$$t = \frac{\pi - \alpha - \beta}{\omega}$$

from (i substituting for ω

$$t = \frac{T(\pi - \alpha - \beta)}{2\pi} \quad \text{(iii}$$

from the diagram $\cos\alpha = \dfrac{\frac{2a}{3}}{a} = \dfrac{2}{3}$

$\alpha = \cos^{-1}\left(\frac{2}{3}\right)$, $\alpha = 0.8412$

$\cos\beta = \dfrac{\frac{3a}{4}}{a} = \dfrac{3}{4}$

$\beta = \cos^{-1}\left(\frac{3}{4}\right)$, $\beta = 0.7227$

substituting for α and β in (iii

$$t = \frac{T(\pi - 0.8412 - 0.7227)}{2\pi}$$

$$= \frac{T}{2\pi}(1.5777) = 0.2510T$$

Ans. time interval is $0.25T$