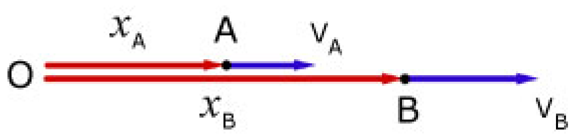
2D Motion : Relative Motion

One dimensional relative velocity(in a line)

Consider two particles A and B at instant t positioned along the x-axis from point O.

Particle A has a displacement $\boldsymbol{x}_A$ from O, and a velocity $\mathbf{V}_A$ along the x-axis. The displacement $\boldsymbol{x}_A$ is a function of time t .

Particle B has a displacement $\boldsymbol{x}_B$ from O, and a velocity $\mathbf{V}_B$ along the x-axis. The displacement $\boldsymbol{x}_B$ is a function of time t .



The velocity $\mathbf{V}_B$ relative to velocity $\mathbf{V}_A$ is written,

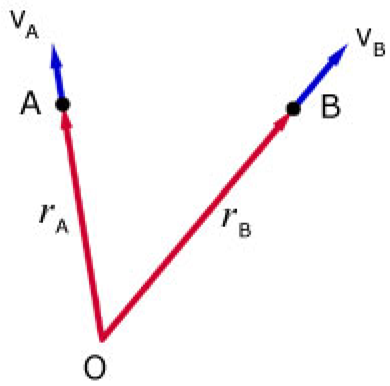
$$_B\mathbf{V}_A = \mathbf{V}_B - \mathbf{V}_A$$

This can be expressed in terms of the derivative of the displacement with respect to time.

$$V_B = \frac{d(\boldsymbol{x}_B)}{dt} \qquad V_A = \frac{d(\boldsymbol{x}_A)}{dt}$$

$$_BV_A = \frac{d(\boldsymbol{x}_B)}{dt} - \frac{d(\boldsymbol{x}_A)}{dt}$$

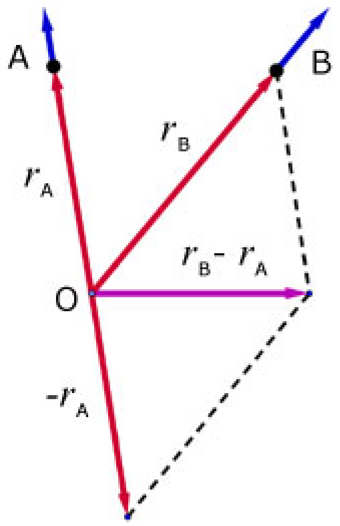
Two dimensional relative position & velocity



Particle A has a displacement $\boldsymbol{r}_A$ from O, and a velocity $\mathbf{V}_A$ along the x-axis. The displacement $\boldsymbol{r}_A$ is a function of time t .

Particle A has a displacement $\boldsymbol{r}_B$ from O, and a velocity $\mathbf{V}_B$ along the x-axis. The displacement $\boldsymbol{r}_B$ is a function of time t .

Relative position

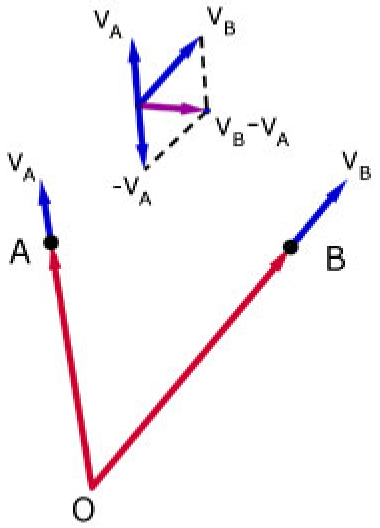


The position of B relative to A at time t is given by the position vector from O, $\boldsymbol{r}_{B\text{-}A}$.

The position vector $\boldsymbol{r}_{B\text{-}A}$ can be written as,

$${}_B\boldsymbol{r}_A = \boldsymbol{r}_B - \boldsymbol{r}_A$$

Relative velocity



Similarly, at time *t* the velocity vector $\mathbf{V}_B$ relative to velocity vector $\mathbf{V}_A$ can be written,

$${}_B\mathbf{V}_A = \mathbf{V}_B - \mathbf{V}_A$$

This can be expressed in terms of the derivative of the displacement with respect to time.

$$V_B = \frac{d(r_B)}{dt} \qquad V_A = \frac{d(r_A)}{dt}$$

$${}_BV_A = \frac{d(r_B)}{dt} - \frac{d(r_A)}{dt}$$

Example #1

If the velocity of a particle P is (9**i** - 2**j**) ms^{-1} and the velocity of another particle Q is (3**i** - 8**j**) ms^{-1} , what is the velocity of particle P relative to Q?

$$\begin{aligned} {}_PV_Q &= V_P - V_Q \\ &= (9\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} - 8\mathbf{j}) \\ &= 9\mathbf{i} - 3\mathbf{i} + 8\mathbf{j} - 2\mathbf{j} \\ &= (9\mathbf{i} - 3\mathbf{i}) + (8\mathbf{j} - 2\mathbf{j}) \\ &= 6\mathbf{i} + 6\mathbf{j} \end{aligned}$$

Ans. velocity of P relative to Q is $(6\mathbf{i} + 6\mathbf{j})\ \text{ms}^{-1}$

Example #2

A particle P has a velocity (4**i** + 3**j**) ms^{-1}. If a second particle Q has a relative velocity to P of (2**i** - 3**j**), what is the velocity of Q?

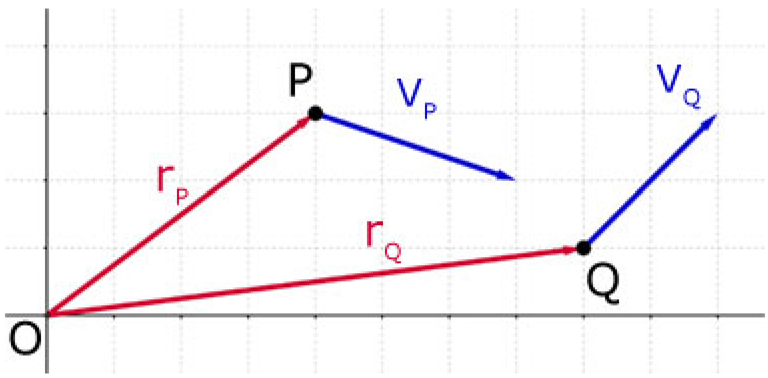
$$
\begin{aligned}
{}_QV_P &= V_Q - V_P \\
V_Q &= {}_QV_P + V_P \\
&= (2\boldsymbol{i} - 3\boldsymbol{j}) + (4\boldsymbol{i} + 3\boldsymbol{j}) \\
&= 2\boldsymbol{i} + 4\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{j} \\
&= 6\boldsymbol{i}
\end{aligned}
$$

$$\underline{\text{Ans. velocity of Q is } (6\boldsymbol{i})\ ms^{-1}}$$

Example #3

A radar station at O tracks two ships P & Q at 0900hours (t=0) .
P has position vector (4**i** + 3**j**) km, with velocity vector (3**i** - **j**) km hr^{-1}.
Q has position vector (8**i** + **j**) km, with velocity vector (2**i** + 2**j**) km hr^{-1}.

i) What is the displacement of P relative to Q at 0900 hours? (ie distance between ships). Answer to 2 d.p.
ii) Write an expression for the displacement of P relative to Q in terms of time *t* .
iii) Hence calculate the displacement of P relative to Q at 1500 hours.



i)

$$\boldsymbol{r}_{\mathrm{P}} = (4\boldsymbol{i} + 3\boldsymbol{j}) \qquad \boldsymbol{r}_{\mathrm{Q}} = (8\boldsymbol{i} + \boldsymbol{j})$$

$$\begin{aligned}\boldsymbol{r}_{\mathrm{P}} - \boldsymbol{r}_{\mathrm{Q}} &= (4\boldsymbol{i} + 3\boldsymbol{j}) - (8\boldsymbol{i} + \boldsymbol{j}) \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} - 8\boldsymbol{i} - \boldsymbol{j} \\ &= 4\boldsymbol{i} - 8\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{j} \\ &= (-4\boldsymbol{i} + 2\boldsymbol{j})\end{aligned}$$

magnitude of the displacement, using Pythagoras,

$$\begin{aligned}\left|\boldsymbol{r}_{\mathrm{P}} - \boldsymbol{r}_{\mathrm{Q}}\right| &= \sqrt{(-4)^2 + (2)^2} \\ &= \sqrt{16+4} = \sqrt{20} = 4.47\end{aligned}$$

Ans. distance between ships at 0900 is 4.47 km.

ii)

$\boldsymbol{r}_{\mathrm{P},0}$ displacement vector of P at time 0

$\boldsymbol{r}_{\mathrm{P},t}$ displacement vector of P at time t

$\boldsymbol{r}_{\mathrm{Q},0}$ displacement vector of Q at time 0

$\boldsymbol{r}_{\mathrm{Q},t}$ displacement vector of Q at time t

$|\mathrm{V}_{\mathrm{P}}t|$ distance travelled by P in time t

$|\mathrm{V}_{\mathrm{Q}}t|$ distance travelled by Q in time t

$$\begin{aligned}\boldsymbol{r}_{\mathrm{P},t} &= \boldsymbol{r}_{\mathrm{P},0} + \mathrm{V}_{\mathrm{P}}t \\ &= (4\boldsymbol{i} + 3\boldsymbol{j}) + (3\boldsymbol{i} - \boldsymbol{j})\end{aligned}$$

$$\begin{aligned}\boldsymbol{r}_{\mathrm{Q},t} &= \boldsymbol{r}_{\mathrm{Q},0} + \mathrm{V}_{\mathrm{Q}}t \\ &= (8\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} - 2\boldsymbol{j})\end{aligned}$$

therefore the displacement of P relative to Q is given by,

$$\begin{aligned} {}_{P,t}\boldsymbol{r}_{Q,t} &= \boldsymbol{r}_{P,t} - \boldsymbol{r}_{Q,t} \\ &= (4\boldsymbol{i} + 3\boldsymbol{j}) + (3\boldsymbol{i} - \boldsymbol{j})t - [(8\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})t] \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{i}t - \boldsymbol{j}t - 8\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{i}t - 2\boldsymbol{j}t \\ &= (4\boldsymbol{i} - 8\boldsymbol{i}) + (3\boldsymbol{j} - \boldsymbol{j}) + 3\boldsymbol{i}t - \boldsymbol{j}t - 2\boldsymbol{i}t - 2\boldsymbol{j}t \\ &= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (3\boldsymbol{i} - 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{j})t \\ &= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})t \end{aligned}$$

$$\underline{{}_{P,t}\boldsymbol{r}_{Q,t} = (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})t}$$

iii) using the result above for 1500 hours(t = 6)

$$\begin{aligned} {}_{P,t}\boldsymbol{r}_{Q,t} &= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})6 \\ &= -4\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{i} - 18\boldsymbol{j} \\ &= -4\boldsymbol{i} + 6\boldsymbol{i} + 2\boldsymbol{j} - 18\boldsymbol{j} \\ &= 2\boldsymbol{i} - 16\boldsymbol{j} \end{aligned}$$

$$\begin{aligned} \left|{}_{P,t}\boldsymbol{r}_{Q,t}\right| &= \sqrt{(2)^2 + (16)^2} \\ &= \sqrt{4 + 256} = \sqrt{260} = 16.12 \end{aligned}$$

$$\underline{\text{Ans. displacement P relative Q at 1500 hours is 16.12 km}}$$

Two dimensional relative acceleration

Similarly, if $\mathbf{a}_A$ and $\mathbf{a}_B$ are the acceleration vectors at A and B at time t, then the acceleration of B relative to A is given by,

$$\mathbf{a}_B = \frac{d(\mathbf{V}_B)}{dt} \qquad \mathbf{a}_A = \frac{d(\mathbf{V}_A)}{dt}$$

$$_B\mathbf{a}_A = \frac{d(\mathbf{V}_B)}{dt} - \frac{d(\mathbf{V}_A)}{dt}$$

$$= \frac{d^2(\mathbf{r}_B)}{dt^2} - \frac{d^2(\mathbf{r}_A)}{dt^2}$$