2D Motion : Circular Motion

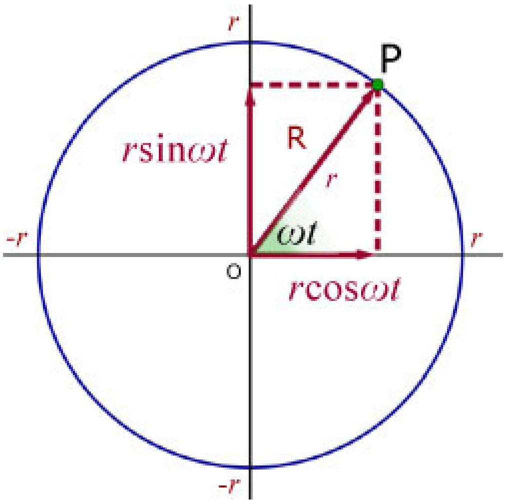
Summary of equations

$$\omega = \frac{\theta}{t} \qquad \omega = \frac{v}{r}$$

$$|\text{radial acceleration}| = \omega^2 r = \frac{v^2}{r}$$

$$|\text{tangential acceleration}| = \frac{dv}{dt}$$

Describing the circle - position vector **R**



i & **j** are unit vectors along the x and y-axis respectively.

The position vector **R** of a particle at P from O, at time t, is given by:

$$\mathbf{R} = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$$

As the position vector **R** rotates anti-clockwise, the particle at P traces out a circle of radius r .

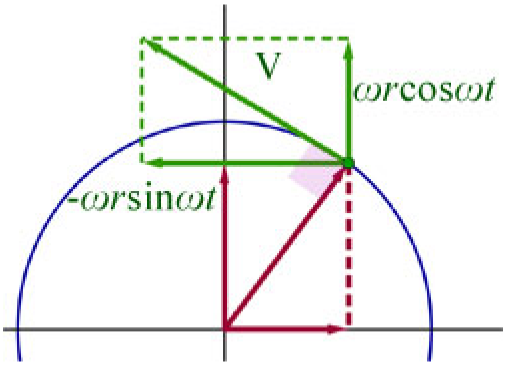
The velocity vector **V**

The velocity vector **V** at an instant is given by differentiating the position vector **R** with respect to t .

Here the unit vectors **i** & **j**, parallel to the x and y-axes, are centred on the particle at P.

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}$$

$$\mathbf{V} = (-\omega r \sin \omega t)\mathbf{i} + (\omega r \cos \omega t)\mathbf{j}$$



the magnitude of the velocity is given by:

$$\begin{aligned}|\mathbf{V}| &= \sqrt{(-\omega r \sin \omega t)^2 + (\omega r \cos \omega t)^2} \\ &= \sqrt{\omega^2 r^2 \sin^2 \omega t + \omega^2 r^2 \cos^2 \omega t} \\ &= \sqrt{\omega^2 r^2 (\sin^2 \omega t + \cos^2 \omega t)}\end{aligned}$$

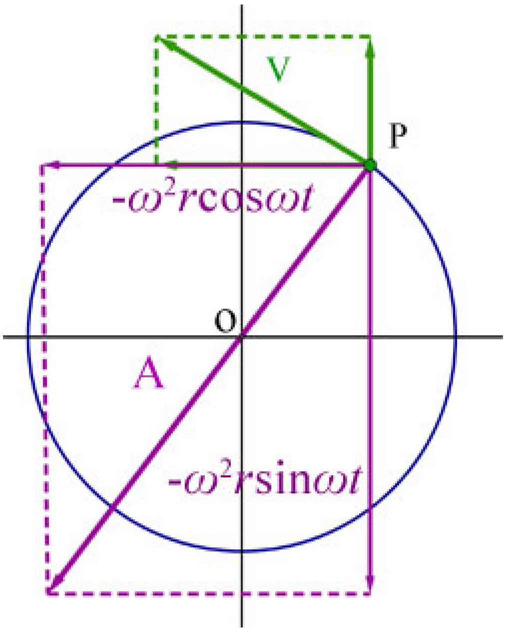
$$\text{but} \quad \sin^2 \omega t + \cos^2 \omega t = 1$$

$$\therefore \quad |\mathbf{V}| = \sqrt{\omega^2 r^2}$$

$$|\mathbf{V}| = \omega r$$

The acceleration vector **A**

The acceleration of the particle at P is given by differentiating **V** with respect to t .



$$\mathbf{A} = \frac{d\mathbf{V}}{dt}$$

$$= (-\omega^2 r\cos\omega t)\mathbf{i} + (-\omega^2 r\sin\omega t)\mathbf{j}$$

but $\mathbf{R} = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$

$\therefore \quad \mathbf{A} = -\omega^2\mathbf{R}$

$\Rightarrow \quad \underline{|\mathbf{A}| = -\omega^2 r}$

Example

A satellite is moving at 2000 ms^{-1} in a circular orbit around a distant moon.
If the radius of the circle followed by the satellite is 1000 km, find:
i) the acceleration of the satellite
ii) the time for the satellite to complete one full orbit of the moon in minutes(2d.p.).

i) a is acceleration, v speed and r orbit radius,

$$a = \frac{v^2}{r}$$

substituting for $v = 2000 \text{ ms}^{-1}$, $r = 1000 \text{ km}(10^6 \text{m})$

$$a = \frac{2000 \times 2000}{1000000} = 4$$

Ans. acceleration of the satellite is 4 ms^{-2}

ii) distance travelled by satellite in one orbit

$$= \text{circumference of orbit circle}$$

$$= 2\pi r = 2\pi \times 10^6$$

$$\text{time for one orbit} = \frac{\text{dist. travelled in one orbit}}{\text{speed}, v}$$

$$= \frac{2\pi \times 10^6}{2 \times 10^3} = 10^3 \pi$$

$$= 3142 \text{ secs.} \equiv 52.3\dot{6} \text{ mins.}$$

Ans. time for one orbit is $52.3\dot{6}$ mins.

Non-uniform circular motion(vertical circle)

A more in-depth treatment of motion in a vertical circle is to be found in 'kinetics/more circular motion'.

Here we look at the more general case of the acceleration component along the circle and the component towards the centre varying.

$$a_{\text{towards centre}} = \frac{v^2}{r}$$

$$a_{\text{along tangent}} = \frac{dv}{dt}$$

Example

A particle starts to move in a circular direction with an angular speed of 5 rad s^{-1} .
The radius of the circle of motion is 4 m, and the angular speed at time t is given by,

$$\omega = 15 - 3t$$

What is,

i) the linear speed of the particle 6 secs. after it starts moving?

ii) the resultant particle acceleration?

(answers to 1 d.p.)

i) $r = 4$ m $t = 6$ secs.

$\omega = \frac{v}{r}$, $v = \omega r$

substituting for $\omega = 15 - 3t$

$v = r(15 - 3t)$

$= 4(15 - 3\times 6) = 4(15 - 18) = 4\times(-3)$

$= -12$

Ans. linear speed of particle is 12.0 ms^{-1} (1d.p.)

ii) tangential acceleration component $= \frac{dv}{dt}$

$\frac{dv}{dt} = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt}$

substituting for $\omega = 15 - 3t$

$\frac{dv}{dt} = r\frac{d(15 - 3t)}{dt} = -3r$

substituting for $r = 4$ m

$\frac{dv}{dt} = -3\times 4 = \underline{-12\ ms^{-2}}$

radial acceleration component $= \frac{v^2}{r}$

$\frac{v^2}{r} = \frac{12\times 12}{4} = \frac{144}{4} = \underline{36\ ms^{-2}}$

resultant acceleration R (using Pythagoras),

$R = \sqrt{(-12)^2 + (36)^2} = \sqrt{144 + 1296} = 37.9$

Ans. resultant acceleration is magnitude 37.9 ms^{-2}