Calculus: Implicit Equations

Explicit equations

Explicit equations are the type we are most familiar with eg y=f(x), $y = 2x^2 + 3x - 5$ etc. where y is expressed in terms of x or some other variable.

Implicit equations

Implicit equations have the structure of being a mix of x and y terms eg $2x^2 + 3xy - 3y^2 = 5$, so y cannot be expressed in terms of x.

The method for solving equations of this type is to regard the whole expression as a function of x and to differentiate both sides of the equation. Any power of y is treated as a 'function of a function', as y is a function of x.

Example #1

find $\frac{dy}{dx}$ for the implicit function:

$$x^3 + 3y^4 - y^2 - 2x = 0$$

$$\frac{d(x^3)}{dx} + \frac{d(3y^4)}{dx} - \frac{d(y^2)}{dx} - \frac{d(2x)}{dx} = 0$$

$$3x^2 + 12y^3\frac{dy}{dx} - 2y\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx}(12y^3 - 2y) = 2 - 3x^2$$

$$\underline{\frac{dy}{dx} = \frac{2 - 3x^2}{12y^3 - 2y}}$$

Example #2

find $\dfrac{dy}{dx}$ for the implicit function:

$$\ln(y) = y\ln(x)$$

for $x > 0$ and $y > 0$

$$\ln(y) = y\ln(x)$$

$$\frac{d(\ln(y))}{dx} = \frac{d(y\ln(x))}{dx} \qquad (i$$

for the expression $\dfrac{d(y\ln(x))}{dx}$

let $u = y$ and $v = \ln(x)$

$$\frac{du}{dy} = \frac{du}{dx}.\frac{dy}{du} = \frac{dy}{dx} \qquad \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d(y\ln(x))}{dx} = y.\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

substituting into $(i$ for $\dfrac{d(y\ln(x))}{dx}$

$$\frac{1}{y}\frac{dy}{dx} = y.\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

rearranging

$$\frac{1}{y}\frac{dy}{dx} - \ln(x)\frac{dy}{dx} = \frac{y}{x}$$

multiplying each side by y

$$\frac{dy}{dx} - y\ln(x)\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx}(1 - y\ln(x)) = \frac{y^2}{x}$$

$$\underline{\frac{dy}{dx} = \frac{y^2}{x(1 - y\ln(x))}}$$

Example #3

find the gradient of the curve:

$x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4,1)$

$$\frac{d(x^2)}{dx} + \frac{d(2xy)}{dx} - \frac{d(2y^2)}{dx} + \frac{d(x)}{dx} = 2$$

$$2x + (2.y + 2x.\frac{dy}{dx}) - 4y.\frac{dy}{dx} + 1 = 0$$

$$\therefore \frac{dy}{dx}(2x - 4y) = -1 - 2x - 2y$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 4y}$$

when $x = -4$ and $y = 1$

$$\frac{dy}{dx} = \frac{-1 - 2(-4) - 2(1)}{2(-4) - 4(1)} = \frac{-1 + 8 - 2}{-8 - 4}$$

$$= \frac{5}{-12} = -\frac{5}{12}$$

gradient at $(-4,1)$ is $-\frac{5}{12}$