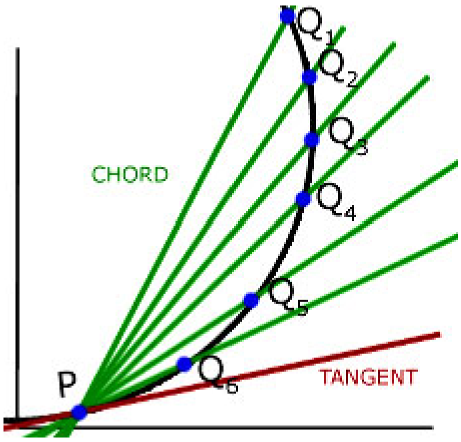
Calculus: the Derivative Formula

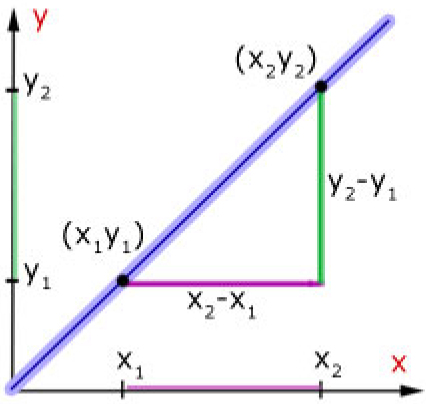
First Principles



To find an expression for the gradient of the tangent at point P on a curve, we must consider lines passing through P and cutting the curve at points Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 ...etc.

As Q approaches P so the gradient of the chord PQ approaches the gradient of the tangent at P.

We can form an expression for the gradient at P by using this concept.

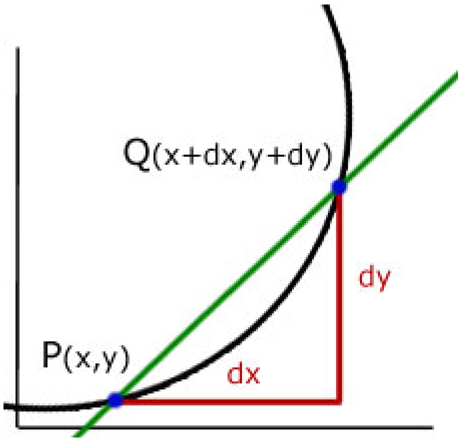


We know from coordinate geometry that:

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

for points (x_1, y_1) and (x_2, y_2)

Consider the coordinates of P to be (x,y) and point Q to be $(x+dx, y+dy)$, where dx and dy are the horizontal and vertical components of the line PQ.



Gradient of the line between points (x,y) and $(x+dx, y+dy)$ is given by :

$$\text{gradient} = \frac{(y+dy)-y}{(x+dx)-x} = \frac{y+dy-y}{x+dx-x} = \frac{dy}{dx}$$

The tangent to the curve = gradient of PQ when the length of PQ is zero and $dx = 0$ and $dy = 0$.

in the limit, as *dx* '<u>approaches zero</u>' the gradient of the curve is said to be *dy/dx*.

If we now replace ***y*** by ***f(x)*** in the expression for gradient, since ***y = f(x)*** i.e. y is a function of x.

$$\lim_{dx \to 0} \frac{(y+dy-y)}{(x+dx-x)}$$

and

$$y = f(x)$$

$$y + dy = f(x + dx)$$

we have:

$$\lim_{dx \to 0} \frac{f(x+dx)-f(x)}{dx}$$

that is,

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x+dx)-f(x)}{dx}$$

Example: find the gradient of $y = 4x^2$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x+dx)-f(x)}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4(x+dx)^2-4(x)^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4(x^2+2xdx+(dx)^2)-4x^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4x^2+8xdx+4(dx)^2-4x^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{8xdx+4(dx)^2}{dx}$$

cancelling by dx

$$\frac{dy}{dx} = 8x + 4dx$$

in the limit when $dx = 0$ this becomes,

$$\frac{dy}{dx} = 8x$$

Without doubt this is a very long winded way to work out gradients. There is a simpler way, by using the Derivation Formula(see further down the page).

Notation This is best described with an example.

If $y = 3x^2$, which can also be expressed as f(x) = $3x^2$, then

the derivative of y with respect to x can be expressed as:

$$\frac{dy}{dx} = 6x \qquad \frac{d(3x^2)}{dx} = 6x \qquad f'(x) = 6x$$

The Derivation Formula

If we have a function of the type $y = \mathrm{k}\,x^n$, where **k** is a

constant, then,

$$\frac{d(\mathrm{k}\,x^{\mathrm{n}})}{dx} = \mathrm{k}\,\mathrm{n}\,x^{\mathrm{n}-1}$$

example #1

Find the gradient to the curve $y = 5\,x^2$ at the point (2,1).

gradient = (5) (2 x^{2-1}) = 10 x^1 = 10 x

gradient at point (2,1) is 10 x 2 = **20**