Calculus: Differential Equations

Definition

An equation containing any **differential coefficients** is called a differential equation.

$$\text{differential coefficients: } \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots$$

The solution of a differential equation is an equation relating x and y and containing **no** differential coefficients.

General & Particular Solution

The **General Solution** includes some unknown constant in the solution of a differential equation.

When some data is given, say the coordinates of a point, then a **Particular Solution** can be formed.

example

differential equation: $\frac{dy}{dx} = 4$

general solution: $y = 4x + c$

(where c is an unknown constant)

if we are given that $x = 3$ when $y = 5$

then $5 = 12 + c$, so $c = -7$

particular solution: $y = 4x - 7$

Example #1

$$\text{find } \frac{d^3y}{dx^3} \text{ when } y = \frac{3}{x}$$

$$y = \frac{3}{x} = 3x^{-1}$$

$$\therefore \frac{dy}{dx} = 3(-1)x^{-2} = -\frac{3}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} = -3(-2)x^{-3} = \frac{6}{x^3}$$

$$\therefore \frac{d^3y}{dx^3} = 6(-3)x^{-4} = \underline{-\frac{18}{x^4}}$$

Example #2

$$\text{given that } \quad y = Ax^2 + B\ln x + C$$

$$\text{show that } \frac{d^2y}{dx^2} = \frac{1}{x}.\frac{dy}{dx}$$

$$y = Ax^2 + B\ln x + C$$

$$\frac{dy}{dx} = 2Ax + \frac{B}{x}$$

$$\frac{d^2y}{dx^2} = 2A - \frac{B}{x^2}$$

$$= \frac{1}{x}\left(2Ax - \frac{B}{x}\right)$$

$$\underline{\frac{d^2y}{dx^2} = \frac{1}{x}.\frac{dy}{dx}}$$

Points of Inflection(Inflexion)

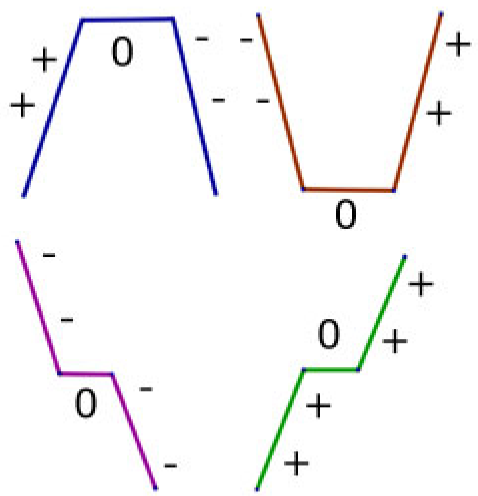
The value of the second derivative can give an indication whether at a point a function has a maximum, minimum or an inflection.These are all called **stationary points**.

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \text{ maximum(-ve)}$$

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \text{ minimum(+ve)}$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \text{ inflection(zero)}$$

A point of inflection has a zero gradient, but the point is not a maximum or a minimum value.



It is where the gradient of a curve decreases(or increases)to zero before increasing(or decreasing)again, but not changing from a negative to a positive value or vice versa.

Example

Find the stationary points of the function:

$y = 3x^4 - 4x^3 - 12x^2 + 5$

$$y = 3x^4 - 4x^3 - 12x^2 + 5$$
$$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x + 1)(x - 2)$$
$$\therefore \; x \text{ has roots } 0,\text{-}1,2$$
$$\frac{d^2y}{dx^2} = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$
$$\text{when } x = 0 \quad \frac{d^2y}{dx^2} = -24 < 0 \quad \therefore \text{max.}(0,10)$$
$$\text{when } x = -1 \quad \frac{d^2y}{dx^2} = 36 > 0 \quad \therefore \text{min.}(-1,5)$$
$$\text{when } x = 2 \quad \frac{d^2y}{dx^2} = 72 > 0 \quad \therefore \text{min.}(2,22)$$