Algebra : Quadratic Equations

Introduction

The general form of a quadratic equation is:

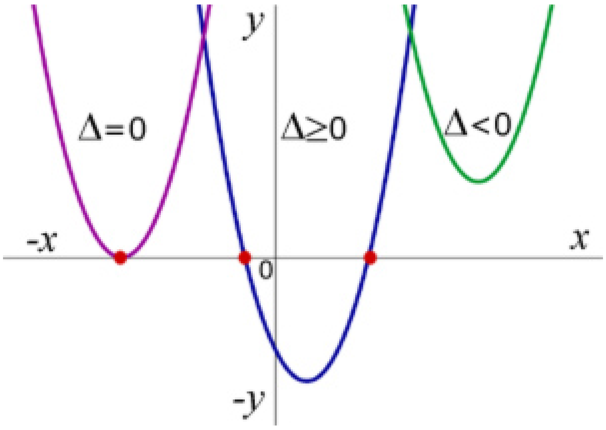
$$ax^2 + bx + c$$

where **a**, **b** & **c** are constants

The expression $b^2 - 4ac$ is called the **discriminant** and given the letter **Δ** (delta).

All quadratic equations have two roots/solutions. These roots are either **REAL**, **EQUAL** or **COMPLEX***.

*complex - involving the square root of –1



real roots	$b^2 - 4ac \geq 0$	$\Delta \geq 0$
equal roots	$b^2 - 4ac = 0$	$\Delta = 0$
complex roots	$b^2 - 4ac < 0$	$\Delta < 0$

Solution by factorising - This is best understood with an example.

solve: $x^2 - 7x + 12 = 0$

You must first ask yourself which two factors when multiplied will give **12 ?**

The factor pairs of **12** are : 1 x 12, 2 x 6 and 3 x 4

You must decide which of these factor pairs added or subtracted will give **7** ?

1 : 12 ...gives 13, 11

2 : 6 .....gives 8, 4

3 : 4 .....gives 7, 1

so $x^2 - 7x + 12 = (x \pm 3)(x \pm 4)$

Which combination when multiplied makes +12 and which when added gives -7?

these are the choices:

(+3)(+4),

(-3)(+4),

(+3)(-4)

(-3)(-4)

Clearly, (-3)(-4) are the two factors we want.

therefore

$$x^2 - 7x + 12 = (x - 3)(x - 4)$$

Now to solve the equation $x^2 - 7x + 12 = 0$.

factorising, as above

$$(x - 3)(x - 4) = 0$$

either

$$(x - 3) = 0$$

or

$$(x-4)=0$$

for the equation to be true.

So the roots of the equation are: $x=3, \quad x=4$

Completing the square

This can be fraught with difficulty, especially if you only half understand what you are doing.

The method is to move the last term of the quadratic over to the right hand side of the equation and to add a number to both sides so that the left hand side can be factorised as the square of two terms.
e.g.

$$x^2-4x-5=0$$
$$x^2-4x=5$$
$$x^2-4x+4=5+4$$
$$x^2-4x+4=9$$
$$(x-2)(x-2)=9$$
$$(x-2)=\pm 3$$
$$x-2=+3\,,\ x=+3+2\,,\ \underline{x=5}$$
$$x-2=-3\,,\ x=-3+2\,,\ \underline{x=-1}$$

However, there is a much neater way of solving this type of problem, and that is by remembering to put the equation in the following form:

$$ax^2+bx+c=0$$
$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$
$$=\left(x+\frac{b}{2a}\right)^2+D$$

using the previous example,

$$x^2 - 4x - 5 = 0 \qquad a = 1, \quad b = -4, \quad c = -5$$

$$x^2 - 4x - 5 = \left(x + \frac{-4}{2\times 1}\right)^2 + D$$

$$x^2 - 4x - 5 = \left(x + (-2)\right)^2 + D$$

$$x^2 - 4x - 5 = (x-2)^2 + D$$

$$x^2 - 4x - 5 = x^2 - 4x + 4 + D$$

$$-5 = 4 + D \ , \ \underline{D = -9}$$

$$(x-2)^2 + D = 0$$

$$(x-2)^2 - 9 = 0$$

$$(x-2)^2 = 9$$

$$(x-2)(x-2) = 9$$

$$(x-2) = \pm 3$$

$$x - 2 = +3 \ , \ x = +3 + 2 \ , \ \underline{x = 5}$$

$$x - 2 = -3 \ , \ x = -3 + 2 \ , \ \underline{x = -1}$$

Using the Formula - the two solutions of quadratic equations in the form

$$ax^2 + bx + c = 0$$

are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

The proof of the formula is by using 'completing the square'.

$$ax^2 + bx + c = 0$$

dividing by a to make the coefficient of x^2 unity

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (i$$

since

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right)$$

$$= \left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \qquad x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

substituting for $x^2 + \frac{b}{a}x$ into (i

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad = \frac{b^2 - 4ac}{4a^2}$$

taking square roots

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example find the two values of x that satisfy the following quadratic equation:

$$2x^2 + 5x - 4 = 0$$

$$a = 2, \quad b = 5, \quad c = -4$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-5 \pm \sqrt{5^2 - 4(2)(-4)}}{2 \times 2} \\
&= \frac{-5 \pm \sqrt{25 + 32}}{4} \\
&= \frac{-5 \pm \sqrt{57}}{4} \\
&= -\frac{5}{4} \pm \frac{\sqrt{57}}{4} \\
&= -1.25 \pm 1.89
\end{aligned}$$

$$x = -1.25 + 1.89, \quad \underline{x = 0.64}$$

$$x = -1.25 - 1.89, \quad \underline{x = -3.11}$$