Algebra: Partial Fractions

some definitions:

Proper Fraction When the degree(index) of the function is higher in the denominator than the numerator.

Improper Fraction When the degree(index) of the function is higher in the numerator than the denominator.

Partial Fractions Factorising the denominator of a proper fraction means that the fraction can be expressed as the sum(or difference) of other proper fractions.

Simple addition/subtraction of algebraic fractions

As with simple fraction arithmetic, a common denominator is found from the denominators of either fraction and the numerators altered to be fractions of the new denominator.

$$\frac{2x}{3y}+\frac{5y}{4x}$$

$$\frac{4x(2x)+3y(5y)}{12xy}$$

$$=\frac{8x^2+15y^2}{12xy}$$

Equations & Identities

Equations are satisfied by **discrete** values of the variable involved.

Example:

$$x^2=9, \qquad x^2-9=0, \qquad (x-3)(x+3)=0$$

$$\underline{x=+3, \qquad x=-3}$$

Identities are satisfied by **any** value of the variable used. Note the equals sign '=' is modified to reflect this.

Example:

$$x^2 - 9 \equiv (x-3)(x+3)$$

When we make partial fractions(below) we are creating an identity from the original expression.

Denominator with only 'linear factors'

By 'linear' we mean that x has a power no higher than '1' . In other words, this method does not work with x^2, x^3, x^4 etc.

For each linear factor of the type:

$$(x-a)$$

there is a partial fraction:

$$\frac{A}{(x-a)}$$

Example:

$$\frac{cx+d}{(x-a)(x-b)} \equiv \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

where x is a variable and A,B,a,b,c,d are constants, where 'a' is not equal to 'b'.

Example #1

$$\frac{5x+3}{(x-1)(x+2)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$\equiv \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\Rightarrow \quad 5x+3 \equiv A(x+2)+B(x-1)$$

making $x=-2$

$$-10+3 = \quad 0 \quad + B(-2-1)$$

$$-7 = -3B$$

$$B = \frac{7}{3}$$

making $x=1$

$$5+3 = A(1+2) + \quad 0$$

$$8 = 3A$$

$$A = \frac{8}{3}$$

$$\therefore \quad \frac{5x+3}{(x-1)(x+2)} \equiv \frac{8}{3(x-1)} + \frac{7}{3(x+2)}$$

Denominator with 'repeated' linear factors

For each 'repeated' linear factor of the type:

$$(x-a)^2$$

there is a partial fraction:

$$\frac{A}{x-a} + \frac{B}{(x-a)^2}$$

Example:

$$\frac{cx+d}{(x-a)^2(x-b)} \equiv \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

Example #1

$$\frac{1}{(x-2)^2(x+3)} \equiv \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x+3)}$$

$$\equiv \frac{A(x-2)(x+3) + B(x+3) + C(x-2)^2}{(x-2)^2(x+3)}$$

$$1 \equiv A(x-2)(x+3) + B(x+3) + C(x-2)^2$$

$$1 \equiv A(x^2 + x - 6) + B(x+3) + C(x^2 - 4x + 4) \quad *$$

putting $x = 2$

$$1 \equiv 0 + B(2+3) + 0$$

$$1 \equiv 5B, \quad \therefore \quad \underline{B = \frac{1}{5}}$$

putting $x = -3$

$$1 \equiv 0 + 0 + C(-5)^2$$

$$1 \equiv 25C, \quad \therefore \quad \underline{C = \frac{1}{25}}$$

equating coefficients of x^2 from *

$$0 = A + C, \quad A = -C, \quad \therefore \quad \underline{A = -\frac{1}{25}}$$

$$\underline{\frac{1}{(x-2)^2(x+3)} \equiv -\frac{1}{25(x-2)} + \frac{1}{5(x-2)^2} + \frac{1}{25(x+3)}}$$

Denominator with a quadratic factor

For each quadratic factor of the type:

$$px^2 + qx + r$$

there is a partial fraction:

$$\frac{Ax + B}{px^2 + qx + r}$$

Example:

$$\frac{cx+d}{(x-a)(px^2+qx+r)} \equiv \frac{A}{x-a} + \frac{Bx+C}{px^2+qx+r}$$

Example #1

$$\frac{3x+2}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\equiv \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$\therefore \quad 3x+2 \equiv A(x^2+1)+(Bx+C)(x-1)$$

$$\equiv Ax^2+A+Bx^2+Cx-Bx-C \quad *$$

putting $x = 1$

$$3+2 = A(1+1) \quad + \quad 0$$

$$5 = 2A, \qquad \underline{A = \frac{5}{2}}$$

putting $x = 0$

$$2 = A + (C)(-1)$$

$$2 = A - C$$

$$2 = \frac{5}{2} - C, \quad \mathrm{C} = \frac{5}{2} - 2, \quad \underline{\mathrm{C} = \frac{1}{2}}$$

from * equating coefficients of x^2

$$0 = A + B$$

$$0 = \frac{5}{2} + B, \qquad \therefore \quad \underline{B = -\frac{5}{2}}$$

$$\frac{3x+2}{(x-1)(x^2+1)} \equiv \frac{5}{2(x-1)} + \frac{\left(-\frac{5}{2}x+\frac{1}{2}\right)}{(x^2+1)}$$

$$\frac{3x+2}{(x-1)(x^2+1)} \equiv \frac{5}{2(x-1)} + \frac{(-5x+1)}{2(x^2+1)}$$

$$\frac{3x+2}{(x-1)(x^2+1)} \equiv \frac{5}{2(x-1)} + \frac{(1-5x)}{2(x^2+1)}$$
