Logarithms

The Laws of Logarithms

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x)^n = n\log_a(x)$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b a}$$

$$\log_a a = 1$$

$$\log_a 1 = 0$$

Proofs

prove $\log_a x - \log_a y = \log_a\left(\dfrac{x}{y}\right)$

let $\log_a x = A$ (i

$\log_a y = B$ (ii

then $x = a^A \quad y = a^B$

$\Rightarrow \quad \dfrac{x}{y} = \dfrac{a^A}{a^B}$

$\dfrac{x}{y} = a^{A-B}$

taking logs each side

$\log_a\left(\dfrac{x}{y}\right) = \log_a a^{A-B}$

$\log_a\left(\dfrac{x}{y}\right) = (A-B)\log_a a$

but $\log_a a = 1$

$\therefore \log_a\left(\dfrac{x}{y}\right) = A - B$

substituting for A and B from (i and (ii

$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

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prove $\log_a x = \dfrac{\log_b x}{\log_b a}$

let $y = \log_a x$

$\Rightarrow \quad x = a^y$

taking logs on both sides to base b

$$\log_b x = \log_b(a^y)$$

$$\log_b x = y\log_b a$$

$$y = \frac{\log_b x}{\log_b a}$$

$$\therefore \quad \log_a x = \frac{\log_b x}{\log_b a}$$

Changing the base

Remember that the change of base occurs in the term where the base is 'x' or some other variable.

Example #1

solve for x $\quad 3\log_2 x - \log_x 2 = 2$

changing $\log_x 2$ to base '2'

$$3\log_2 x - \frac{\log_2 2}{\log_2 x} = 2$$

multiplying both sides by $\log_2 x$

$$3(\log_2 x)^2 - \log_2 2 = 2(\log_2 x)$$

rearranging

$$3(\log_2 x)^2 - 2(\log_2 x) - \log_2 2 = 0$$

but $\log_2 2 = 1$

$$\therefore \quad 3(\log_2 x)^2 - 2(\log_2 x) - 1 = 0$$

$$(3(\log_2 x) + 1)(\log_2 x - 1) = 0$$

$$\log_2 x = -\tfrac{1}{3} \quad \text{or} \quad \log_2 x = 1$$

$$\Rightarrow \quad x = 2^{-\frac{1}{3}} \quad \text{or} \quad x = 2^1 = 2$$

Simultaneous equations

'Substitution' simultaneous equations are common problems. First find what x is in terms of y. Then substitute for x in the other equation. Solve for y.

Example #1

given that $\log_2(x-3y+2)=0$ (i

and $\log_2(x+1)-1=2\log_2 y$ (ii

find x and y

$$\log_2(x-3y+2)=0$$

$\Rightarrow$ $x-3y+2=2^0$

but $2^0=1$

$\therefore$ $x-3y+2=1$

$\underline{x=3y-1}$ (iii

substituting for x into (ii

$$\log_2(x+1)-1=2\log_2 y$$
$$\log_2(3y-1+1)-1=2\log_2 y$$
$$\log_2 3y-1=2\log_2 y$$
$$\log_2 3y-2\log_2 y=1$$
$$\log_2\left(\frac{3y}{y^2}\right)=1$$

$\Rightarrow$ $\frac{3}{y}=2^1$, $2y=3$, $\underline{y=1\tfrac{1}{2}}$

substituting for y in (iii

$$x=3y-1$$
$$=3(1\tfrac{1}{2})-1$$
$$=4\tfrac{1}{2}-1=3\tfrac{1}{2}$$

$\therefore$ $\underline{\text{when } x=3\tfrac{1}{2},\quad y=1\tfrac{1}{2}}$

Variable in the index

Take logs on both sides. Move the indices infront of the logs. Expand the equation. Collect x-terms to the left. Sum the numbers to the right. These problems can be tricky with the amount of arithmetic involved. So make sure you write everything down to make checking your working easier.

Example #1

solve for x to 3 d.p. $2^{2x+1}\times 3^{x-1} = 6^x$

$$\Rightarrow \quad \log_{10} 2^{2x+1} + \log_{10} 3^{x-1} = \log_{10} 6^x$$
$$\therefore \quad (2x+1)\log_{10} 2 + (x-1)\log_{10} 3 = x\log_{10} 6$$
$$(2x+1)(0.3010) + (x-1)(0.4771) = x(0.7782)$$
$$(0.602)x + 0.3010 + (0.4771)x - 0.4771 = (0.7782)x$$
$$(0.602)x + (0.4771)x - (0.7782)x = 0.4771 - 0.3010$$
$$(0.3009)x = 0.1761$$
$$x = 0.58524$$
$$\text{to 3 d.p.} \quad \underline{x = 0.585}$$