Algebra : Iteration

Introduction

Repeatedly solving an equation to obtain a result using the result from the previous calculation, is called '**iteration**'. The procedure is used in mathematics to give a more accurate answer when the original data is only approximate.

Problems usually involve finding the root of an equation when only an approximate value is given for where the curve crosses an axis.

Direct/Fixed Point Iteration

method:

1. rearrange the given equation to make the highest power of x the subject

2. find the power root of each side, leaving x on its own on the left

3. the LHS x becomes x_{n+1}

4. the RHS x becomes x_n

The equation is now in its iterative form.

We start by working out x_2 from the given value x_1 .
x_3 is worked out using the value x_2 in the equation.
x_4 is worked out using the value x_3 and so on.

Example

Find correct to 3 d.p. a root of the equation

$$f(x) = x^3 - 2x + 3$$

given that there is a solution near $x = -2$

$$x^3 - 2x + 3 = 0$$
$$x^3 = 2x - 3$$
$$x = \sqrt[3]{2x-3}$$
$$x = (2x-3)^{\frac{1}{3}}$$

$$\Rightarrow \quad x_{n+1} = (2x_n - 3)^{\frac{1}{3}}$$

$$x_1 = -2$$
$$x_2 = (2x_1 - 3)^{\frac{1}{3}} = (2(-2) - 3)^{\frac{1}{3}} = -1.9129$$
$$x_3 = (2x_2 - 3)^{\frac{1}{3}} = (2(-1.9129) - 3)^{\frac{1}{3}} = -1.8969$$
$$x_4 = (2x_3 - 3)^{\frac{1}{3}} = (2(-1.8969) - 3)^{\frac{1}{3}} = -1.8940$$
$$x_5 = (2x_4 - 3)^{\frac{1}{3}} = (2(-1.8940) - 3)^{\frac{1}{3}} = -1.8934$$
$$x_6 = (2x_5 - 3)^{\frac{1}{3}} = (2(-1.8934) - 3)^{\frac{1}{3}} = -1.8933$$
$$x_7 = (2x_6 - 3)^{\frac{1}{3}} = (2(-1.8933) - 3)^{\frac{1}{3}} = -1.8933$$

$$\Rightarrow \quad \underline{x = 1.893} \quad (3 \text{ d.p.})$$

Iteration by Bisection

method:

1. reduce the interval where the root lies into two equal parts
2. decide in which part the solution resides
3. repeat the process until a consistent answer is achieved for the degree of accuracy required

Example

Find correct to 3 d.p. a root of the equation

$$f(x) = 2x^2 - 2x + 7$$

given that there is a solution near $x = -2$

$$f(x) = 2x^3 - 2x + 7$$

$$f(-2) = 2(-2)^3 - 2(-2) + 7 = -5$$
$$f(-1) = 2(-1)^3 - 2(-1) + 7 = +7$$
$$\Rightarrow \qquad -2 < \text{root} < -1$$
$$f(-1.5) = 2(-1.5)^3 - 2(-1.5) + 7 = +6.625$$
$$\Rightarrow \qquad -2 < \text{root} < -1.5$$
$$f(-1.75) = 2(-1.75)^3 - 2(-1.75) + 7 = -0.21875$$
$$\Rightarrow \qquad -1.75 < \text{root} < -1.5$$
$$f(-1.625) = -0.21875$$
$$\Rightarrow \qquad -1.75 < \text{root} < -1.625$$
$$f(-1.688) = +0.7566$$
$$\Rightarrow \qquad -1.75 < \text{root} < -1.688$$
$$f(-1.719) = +0.2788$$
$$\Rightarrow \qquad -1.75 < \text{root} < -1.719$$
$$f(-1.7345) = +0.0325$$
$$\Rightarrow \qquad -1.75 < \text{root} < -1.7345$$
$$f(-1.7423) = +0.0925$$
$$\Rightarrow \qquad -1.7423 < \text{root} < -1.7345$$
$$f(-1.7384) = +0.0302$$
$$\Rightarrow \qquad -1.7423 < \text{root} < -1.7384$$
$$\therefore \quad \text{to 3 s.f.} \qquad -1.74 < \text{root} < -1.74$$

$$\underline{x = -1.74}$$

Newton-Raphson Method

This uses a tangent to a curve near one of its roots and the fact that where the tangent meets the x-axis gives an approximation to the root.

The iterative formula used is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

Find correct to 3 d.p. a root of the equation

$$f(x) = 2x^2 + x - 6$$

given that there is a solution near $x = 1.4$

$f(x) = 2x^2 + x - 6 \qquad f'(x) = 4x + 1$

n	x_n	$f(x)$	$f'(x)$	x_{n+1}
1	1.4	-0.68	6.6	1.503
2	1.503	0.0210	7.012	1.4731
3	1.4731	-0.1869	6.8924	1.5002
4	1.5002	0.0014	7.001	1.500
5	1.500	0.0	7.0	1.500
6	1.500			

$\Rightarrow \quad \underline{x = 1.500} \quad$ (3 d.p.)