Calculus: The Chain Rule

The Chain Rule Equation

This is a way of differentiating a function of a function.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Example #1

differentiate $(3x+3)^3$

let $y=(3x+3)^3$ and $t = 3x+3$

then $y = t^3$

$\frac{dt}{dx} = 3$, $\frac{dy}{dt} = 3t^2$

using the Chain Rule

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, $\therefore \frac{dy}{dx} = 3t^2 . 3 = 9t^2$

$$\frac{d\left\{(3x+3)^3\right\}}{dx} = 9(3x+3)^2 = 9(3)(x+1)(3)(x+1)$$

$$= 81(x+1)^2$$

Example #2

differentiate $(x^2+5x)^6$

let $y=(x^2+5x)^6$ and $t = x^2+5x$

then $y = t^6$

$\frac{dt}{dx} = 2x+5$, $\frac{dy}{dt} = 6t^5$

using the Chain Rule

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, $\frac{dy}{dx} = 6t^5 . 2x+5$

$$\frac{d\left\{(x^2+5x)^6\right\}}{dx} = 6(x^2+5x)^5(2x+5)$$

$$= 6(2x+5)(x^2+5x)^5$$

Rates of change

The Chain Rule is a means of connecting the rates of change of dependent variables.

Example #1

If air is blown into a spherical balloon at the rate of 10 cm[3] how quickly will the radius grow?

if the radius of the balloon is r

then the volume $V = \frac{4}{3}\pi r^3$

and $\frac{dV}{dr} = 4\pi r^2$

the rate of change of volume with time

is given by: $\frac{dV}{dt} = 10\text{cm}^3 / \text{sec.}$

using the Chain Rule

$\frac{dV}{dt} = \frac{dV}{dr} . \frac{dr}{dt}$ and $\frac{dV}{dr} = 4\pi r^2$

$$\therefore \frac{dr}{dt} = \frac{dV}{dt} . \frac{1}{\frac{dV}{dr}} = \frac{dV}{dt} . \frac{dr}{dV} = 10 . \frac{1}{4\pi r^2}$$

$$= \frac{5}{2\pi r^2}$$

i.e. rate of change of radius is $\frac{5}{2\pi r^2}$ cm/sec.

Example #2

A spherical raindrop is formed by condensation. In an interval of 10 sec. its volume increases at a constant rate from $0.010mm^3$ to $0.500mm^3$.

Find the rate at which the surface area of the raindrop is increasing, when its radius is 1.0mm

radius r mm

volume V is given by: $V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dr} = \frac{4\pi}{3}.3r^2 = 4\pi r^2$

also, area A is given by: $A = 4\pi r^2$

$\therefore \quad \frac{dA}{dr} = 4\pi.2r = 8\pi r$

vol. increases at a constant rate by:

$0.5 - 0.010 = 0.490$ mm^3 in 10 sec.

so $\frac{dV}{dt} = \frac{0.49}{10} = 0.049$ mm^3.s^{-1}.

we are required to find $\frac{dA}{dt}$ when $r = 1.0$mm

using the Chain Rule, $\frac{dA}{dt} = \left(\frac{dA}{dV}\right).\frac{dV}{dt}$

$$= \left(\frac{dA}{dr}.\frac{dr}{dV}\right).\frac{dV}{dt}$$

$$\frac{dA}{dt} = \left(8\pi r.\frac{1}{4\pi r^2}\right)0.049 = \left(\frac{2\times 0.049}{r}\right) = \frac{0.098}{r}$$

when $r = 1.0$mm, $\frac{dA}{dt} = \frac{0.098}{1} = 0.098$mm^2.s^{-1}.

$\therefore$ surface area, for a radius of 1mm, increases by 0.098mm^2.s^{-1}.