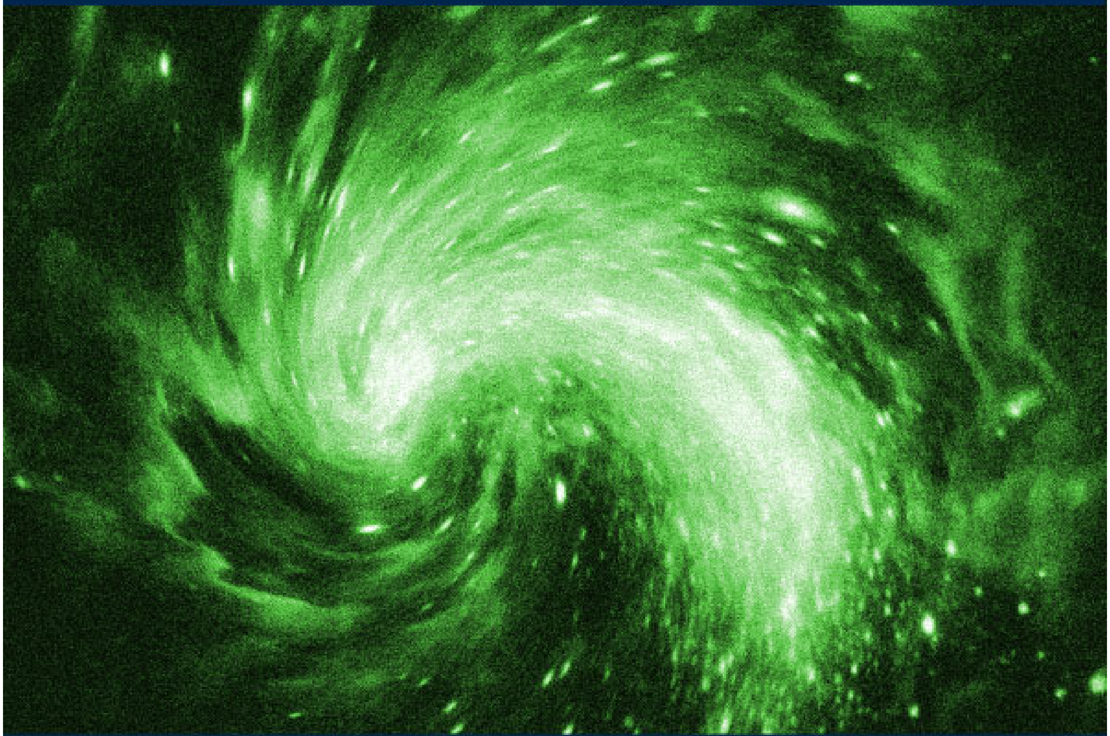
A-LEVEL MATHS TUTOR

Pure Maths



PART FIVE

SEQUENCES & SERIES

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The Sigma Notation

Introduction

An ordered set of numbers obeying a simple rule is called a **sequence**.

2, 4, 6, 8, 10...

17, 22, 27, 32, 37... etc.

A **series** or progression is when the terms of a sequence are considered as a sum.

2 + 4 + 6 + 8 + 10 +...

17 + 22 + 27 + 32 + 37 + ... etc.

Sigma Notation

Instead of writing long expressions like

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots + n^3$$

we are able to write:

$$\sum m^3$$

which means ' the sum of all terms like m^3 '

To show where a series begins and ends, numbers are placed above and below the sigma symbol. These are equal to the value of the variable, 'm' in this case, taken in order.

Hence

$$\sum_{1}^{n} m^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3$$

more examples

$$\sum_{2}^{5} m(m-1) = 2(2-1) + 3(3-1) + 4(4-1) + 5(5-1)$$

$$\sum_{3}^{7} m^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

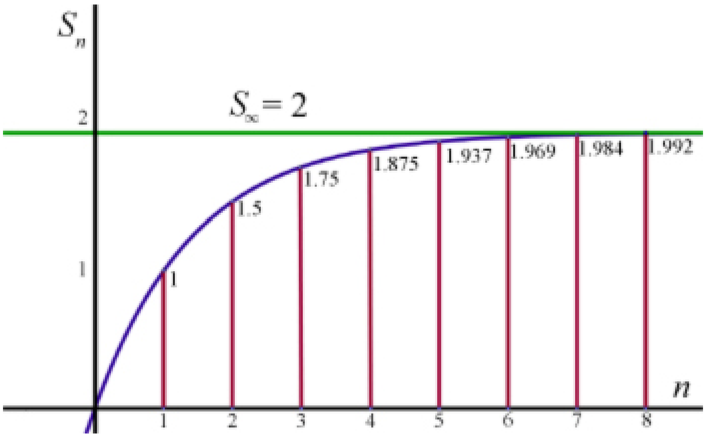
$$\sum_{4}^{9} \frac{1}{m} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$$

Convergence

This concerns geometrical progressions that as the number of terms increase, the value of the sum approaches one specific number. This number is called **the sum to infinity**.

Look at this example. As the number of terms(n) increases, the sum of the progression (S_n)approaches the number 2.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots + \frac{1}{2^{n-1}}$$



You can find out more about convergent series in the topic ‘geometrical progressions’.

Recurrence

Recurrence is when there is some mathematical relation between consecutive terms in a sequence.

The Fibonacci series is a good example of this. The numbers of the series are made up by adding the two previous numbers.

$$0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 \ldots$$

Arithmetical Series

Arithmetical series structure

An arithmetical series starts with the first term, usually given the letter '**a**'. For each subsequent term of the series another term is added. This is a multiple of the letter '**d**' called 'the **common difference**'.

So the series has the structure:

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (l - d) + l$$

where $\mathbf{S_n}$is the sum to '**n**' terms, the letter '**l**' is the last term.

The common difference 'd' is calculated by subtracting any term from the subsequent term.

The **nth term**(sometimes called the 'general term')is given by:

$$a + (n-1)d$$

Proof of the sum of an arithmetical series

The sum of an arithmetical series is found by adding two identical series together, but putting the second series in reverse order.

$$
\begin{aligned}
S_n &= a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l \\
\underline{S_n} &\underline{= l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a} \\
2S_n &= (a+l) + (a+l) + (a+l) + \ldots \quad + (a+l) + (a+l) + (a+l)
\end{aligned}
$$

noting that there are still n terms on the RHS

$$\therefore\ 2S_n = n(a+l)$$

$$\Rightarrow \qquad S_n = \frac{n(a+l)}{2} \qquad (i$$

since the last term l is given by:

$$l = a + (n-1)d$$

substituting for l in (i

$$S_n = \frac{n(a + a + (n-1)d)}{2}$$

$$\therefore \qquad S_n = \frac{n}{2}(2a + (n-1)d)$$

Example #1

In an arithmetical progression the 8th term is 23 and the 11th term is 4 times the 3rd term.

Find the 1st term, the common difference and the sum of the first 10 terms.

let the 1st term be a and the common difference d

then the 8th terms is given by:

$$a + 7d = 23 \qquad \text{(i}$$

the 11th term is $(a+10d)$, the 3rd term is $(a+2d)$

$$\therefore \quad a + 10d = 4(a + 2d)$$

$$a + 10d = 4a + 8d$$

$$10d - 8d = 4a - a$$

$$2d = 3a, \quad \underline{a = \tfrac{2}{3}d} \qquad \text{(ii}$$

substituting into (i above for a

$$a + 7d = 23$$

$$\tfrac{2}{3}d + 7d = 23$$

$$2d + 21d = 69$$

$$23d = 69, \quad \underline{d = 3}$$

substituting into (ii for d

$$a = \tfrac{2}{3}(3), \quad \underline{a = 2}$$

$\therefore$ the first term is 2 and the common difference is 3

sum of the first 10 terms is given by:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_{10} = \frac{10}{2}(2(2) + (10-1)3)$$

$$= 5(4 + 9 \times 3)$$

$$= 5 \times 31$$

$$= 155$$

sum of the first 10 terms is 155

Example #2

The sum of terms of an arithmetic progression is 48.

If the first term is 3 and the common difference is 2, find the number of terms.

since

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

and $\quad a = 3, \quad d = 2, \quad S_n = 48$

the sum of the first n terms is given by:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$48 = \frac{n}{2}(2 \times 3 + (n-1)2)$$

$$96 = 6n + 2n^2 - 2n$$

$$2n^2 + 4n - 96 = 0$$

$$n^2 + 2n - 48 = 0$$

$$(n-6)(n+8) = 0$$

hence $\quad n = 6 \quad$ or $\quad n = -8$

$\therefore$ the number of terms is 6

Arithmetic Mean

This is a method of finding a term sandwiched between two other terms. The required term is calculated by taking an average of the term before and the term after.

So if we have a sequence of terms: **a b c** and **a** and **c** are known. Then term **b** is given by:

$$b = \frac{a+c}{2}$$

Example

If the 3rd term of an arithmetical progression is 7 and the 5th term is 11, what is the 4th term?

let the 3rd term be 'a' and the 5th term 'c'

then the 4th term 'b' is given by:

$$\begin{aligned} b &= \frac{a+c}{2} \\ &= \frac{7+11}{2} \\ &= \frac{18}{2} \end{aligned}$$

$$\underline{b = 9}$$

Geometrical Series

Geometrical series structure

A geometricall series starts with the first term, usually given the letter '**a**'. For each subsequent term of the series the first term is multiplied by another term. The term is a multiple of the letter '**r**' called 'the **common ratio** '.

So the series has the structure:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1}$$

where $\mathbf{S_n}$is the sum to '**n**' terms, the letter '**l**' is the last term.

The common ratio 'r' is calculated by dividing any term by the term before it.

The **nth term**(sometimes called the 'general term')is given by:

$$ar^{n-1}$$

Proof of the sum of a geometrical series

$$\begin{aligned}
& S_n = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} \\
\therefore \quad & rS_n = \quad ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n \\
& \text{subtracting} \\
& S_n - rS_n = a - ar^n \\
& S_n(1-r) = a(1-r^n) \\
\therefore \quad & S_n = a\left(\frac{1-r^n}{1-r}\right)
\end{aligned}$$

NB an alternative formula for r > 1 , just multiply numerator & denominator by -1

Example #1

In a geometrical progression the sum of the 3rd & 4th terms is 60 and the sum of the 4th & 5th terms is 120.

Find the 1st term and the common ratio.

with the 1st term a and the common difference r

$$ar^2 + ar^3 = 60$$

$$ar^3 + ar^4 = 120$$

factorising

$$ar^2(1+r) = 60$$

$$ar^3(1+r) = 120$$

dividing

$$\frac{1}{r} = \frac{1}{2}, \quad \underline{r = 2}$$

substituting $r = 2$ into $ar^2(1+r) = 60$

$$a(4)(1+2) = 60$$

$$12a = 60$$

$$\underline{a = 5}$$

<u>the 1st term is 5 and the common ratio is 2</u>

Example #2

What is the smallest number of terms of the geometrical progression

2 + 6 + 18 + 54 + 162 ...

that will give a total greater than 1000?

from the series, $a = 2$ and $r = 3$

using the expression for the sum of a G.P.

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

$$S_n = 2\left(\frac{3^n - 1}{3 - 1}\right)$$

$$S_n = 3^n - 1$$

making $S_n = 1000$

$$\Rightarrow \qquad 3^n - 1 = 1000$$

taking logs to base 10 each side

$$n\log_{10}3 = \log_{10}1000$$

$$n\log_{10}3 = \log_{10}10^3$$

$$n\log_{10}3 = 3\log_{10}10$$

but $\log_{10}10 = 1$

$$\therefore \qquad n\log_{10}3 = 3$$

$$n = \frac{3}{\log_{10}3}$$

$$= \frac{3}{0.47712}$$

$$= 6.2877$$

$\therefore$ for a total exceeding 1000 $\quad n = 7$

Geometric Mean

This is a method of finding a term sandwiched between two other terms.

So if we have a sequence of terms: **a b c** and **a** and **c** are known. The ratio of successive terms gives the common ratio. Equating these**:**

$$\frac{b}{a} = \frac{c}{b}$$
$$b^2 = ac$$
$$\underline{b = \sqrt{ac}}$$

Example

If the 4th term of a geometrical progression is 40 and the 6th is 160, what is the 5th term?

using the expression for the mean,
for series a, b, c

$$\begin{aligned} b &= \sqrt{ac} \\ &= \sqrt{40 \times 160} \\ &= \sqrt{6400} \\ &= 80 \end{aligned}$$

the 5th term is 80

Sum to infinity

This concerns geometrical progressions that as the number of terms increase, the value of the sum approaches one specific number. This number is called **the sum to infinity**.

In this example as 'n' increases the sum approaches 2.

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots+\frac{1}{2^{n-1}}$$

$$\text{for} \quad -1<r<1 \qquad S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$\text{from the series} \quad a=1, \quad r=\tfrac{1}{2}$$

$$\therefore \qquad S_n = 1\left(\frac{1-\frac{1}{2}^n}{1-\frac{1}{2}}\right)$$

$$S_n = 2(1-(\tfrac{1}{2})^n)$$

$$\text{as} \quad n\rightarrow\infty, \quad (\tfrac{1}{2})^n \rightarrow 0$$

$$\therefore \qquad S_n \approx 2(1-0)$$

$$\underline{S_n \approx 2}$$

So if the term r^n tends to zero, with increasing n the equation for the sum to n terms changes:

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \quad \text{becomes} \quad S_n = \frac{a}{1-r}$$

<u>Example</u>

Express 0.055555... as a fraction.

$0.0555555\ldots$ or $0.0\dot{5}$ may be written

$$\frac{5}{100}+\frac{5}{1000}+\frac{5}{10000}+\frac{5}{100000}+\frac{5}{1000000}+\ldots$$

where $a=\dfrac{5}{100}$ and $r=\dfrac{1}{10}$

using the equation for S_n to ∞

$$S_n=\frac{a}{1-r}$$

$$\Rightarrow \qquad S_\infty=\frac{5}{100}\left(\frac{1}{1-\frac{1}{10}}\right)$$

$$=\frac{5}{100}\left(\frac{1}{\frac{9}{10}}\right)$$

$$=\frac{5}{100}\left(\frac{10}{9}\right)$$

$$=\frac{5}{10}\left(\frac{1}{9}\right)=\frac{1}{2}\times\frac{1}{9}=\frac{1}{18}$$

$0.\dot{5}$ may be written as the fraction $\dfrac{1}{18}$

Notes