A-LEVEL MATHS TUTOR

Mechanics

PART THREE

KINETICS

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Contents

Newton's Laws

Newton's Laws of Motion

1. A body will remain at rest or travel at uniform linear velocity unless acted upon by an external force.

2. The rate of change of linear momentum is proportional to the applied force and acts in the same direction as the force.

3. The forces of two bodies on each other are equal and directed in opposite directions.

Definition of momentum

Momentum(**P**) is a vector quantity equal in magnitude to the product of mass and velocity.

Note, mass(m) is a scalar quantity, while velocity(**v**) is a vector quantity.

$$\mathbf{P} = m\mathbf{v}$$

m (kg)   **v**(ms^{-1})   **P** ($kg.\ ms^{-1}$)

The letter 'p' in small case is designated to represent pressure.

Theory

If we consider a force **F** acting on a mass m with velocity **v**, the Second law may be represented by the proportionality:

$$\mathbf{F} \propto \frac{d(m\mathbf{v})}{dt}$$

$$\mathbf{F} = k\,\frac{d(m\mathbf{v})}{dt}$$

$$= k\,m\,\frac{d(\mathbf{v})}{dt}$$

$$\text{but acceleration} \qquad \mathbf{a} = \frac{d(\mathbf{v})}{dt}$$

$$\Rightarrow \quad \mathbf{F} = km\mathbf{a}$$

$$\text{making} \quad \mathbf{F} = 1\ \text{Newton}, \quad m = 1\ \text{kg}. \quad \mathbf{a} = 1\,\text{ms}^{-2}$$

$$\Rightarrow \quad k = 0$$

$$\underline{\Rightarrow \quad \mathbf{F} = m\mathbf{a} \qquad \text{force} = \text{mass} \times \text{acceleration}}$$

The Newton N

As seen from the theory relating to the Second Law, to get rid of the constant of proportionality each quantity is made unity.

So we come to our definition of a Newton:

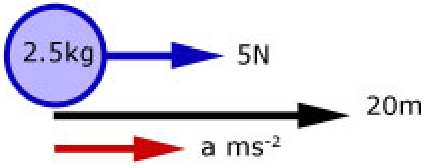
A **Newton** is the force that when applied to a **1 kg mass** will give it an **acceleration of 1 ms^{-2}**.

Linear acceleration

Here the mass is either stationary and is accelerated by a force in a straight line or is initially moving at constant velocity before the force is applied.

Example #1

A 5N force acts on a 2.5kg mass, making it accelerate in a straight line.
i) What is the acceleration of the mass?
ii) How long will it take to move the mass through 20m?
(Answer to 2 d.p.)



i)

$$F = 5\text{N} \qquad m = 2.5\text{kg} \qquad s = 20\text{m} \qquad u = 0\text{ms}^{-1}$$

$$\text{using} \quad F = ma$$

$$\Rightarrow \quad a = \frac{F}{m} = \frac{5}{2.5} = 2$$

$$\underline{\text{Ans. acceleration is } 2 \text{ ms}^{-1}}$$

ii)

$$\text{using} \qquad s = ut + \tfrac{1}{2}at^2$$

$$u = 0\text{ms}^{-1} \quad \Rightarrow \quad s = \tfrac{1}{2}at^2$$

$$t^2 = \frac{2s}{a}$$

$$t = \sqrt{\frac{2s}{a}}$$

$$s = 20\text{m} \qquad a = 2\text{ms}^{-2}$$

$$t = \sqrt{\frac{2 \times 20}{2}} = \sqrt{20} = 4.47$$

$$\underline{\text{Ans. time for mass to move 20 m is 4.47 secs.}}$$

Example #2

A force causes a 3kg mass to accelerate. If the velocity of the mass at time *t* is given by:

$$\mathbf{v} = 2t\mathbf{i} - 3t^2\mathbf{j} + 4t^3\mathbf{k}$$

what is the magnitude of the force when *t* = 5 secs. ?

acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt}$

$$\frac{d\mathbf{v}}{dt} = 2\mathbf{i} - 6t\mathbf{j} + 12t^2\mathbf{k}$$

using $\mathbf{F} = m\mathbf{a}$

$$= m(2\mathbf{i} - 6t\mathbf{j} + 12t^2\mathbf{k})$$

substituting for $m = 3$ kg, $t = 5$ secs.

$$\mathbf{F} = 3(2\mathbf{i} - 30\mathbf{j} + 300\mathbf{k})$$

$$= 6\mathbf{i} - 90\mathbf{j} + 900\mathbf{k})$$

hence the magnitude of $\mathbf{F}$ is given by:

$$|\mathbf{F}| = \sqrt{(6)^2 + (-90)^2 + (900)^2}$$

$$= \sqrt{36 + 8100 + 810000}$$

$$= \sqrt{818136} = 904.51$$

<u>Ans. when $t = 5$ secs. applied force is 904.51 N</u>

Linear retardation

Here the mass is already moving at constant velocity in a straight line before the force is applied, opposing the motion.

Example #1

A 4 kg mass travelling at constant velocity 15 ms^{-1} has a 10 N force applied to it against the direction of motion.
i) What is the deceleration produced?
ii) How long will it take before the mass is brought to rest?

i)

$$u = 15 \text{ ms}^{-1} \qquad m = 4 \text{ kg} \qquad F = 10 \text{ N}$$

$$F = ma$$

$$a = \frac{F}{m}$$

$$a = \frac{10}{4} = 2.5$$

$\underline{\text{Ans. deceleration is } 2.5 \text{ ms}^{-1}}$

ii)

$$v = 0$$

$$v = u - at$$

$$0 = u - at$$

$$at = u$$

$$t = \frac{u}{a} = \frac{15}{2.5} = 6$$

$\underline{\text{Ans. mass brought to rest in 6 secs.}}$

Example #2

A sky diver with mass 80kg is falling at a constant velocity of 70 ms^{-1} . When he opens his parachute he experiences a constant deceleration of 3g for 2 seconds.
i) What is the magnitude of the decelerating force?
ii) What is his rate of descent at the end of the 2 seconds deceleration?

i)

$$m = 80 \text{ kg} \qquad u = 70 \text{ ms}^{-1} \quad a = 3g$$

$$\begin{aligned} F &= ma \\ &= 80 \times 3 \times 9.8 \\ &= 2352 \end{aligned}$$

$$\underline{\text{Ans. decelerating force is } 2352 \text{ N}}$$

ii)

$$u = 70 \text{ ms}^{-1} \quad a = 3 \times 9.8 = 29.4 \text{ ms}^{-2} \qquad t = 2 \text{ secs.}$$

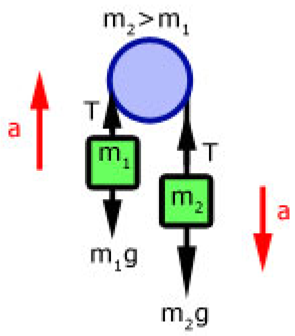
$$\begin{aligned} v &= u - at \\ &= 70 - (29.4) \times (2) \\ &= 11.2 \end{aligned}$$

$$\underline{\text{Ans. final rate of descent is } 11.2 \text{ ms}^{-1}}$$

Connected Particles

Pulleys

Problems involve two weights either side of a pulley. The heavier weight pulls on the lighter causing both to accelerate in one direction with a common acceleration.



calculation of acceleration 'a'

$$m_2 > m_1$$

$$T - m_1 g = m_1 a$$

$$\underline{m_2 g - T = m_2 a}$$

$$m_2 g - m_1 g = (m_1 + m_2)a$$

$$a = \frac{(m_2 - m_2)}{(m_1 + m_2)} g$$

calculation of tension 'T'

$$m_2 > m_1$$

$$T - m_1 g = m_1 a$$

$$a = \frac{T - m_1 g}{m_1} \qquad (i$$

$$m_2 g - T = m_2 a$$

$$a = \frac{m_2 g - T}{m_2} \qquad (ii$$

equating (i and (ii

$$\frac{T - m_1 g}{m_1} = \frac{m_2 g - T}{m_2}$$

$$m_2 T - m_1 m_2 g = m_1 m_2 g - m_1 T$$

$$m_2 T + m_1 T = m_1 m_2 g + m_1 m_2 g$$

$$T\left(m_2 + m_1\right) = 2 m_1 m_2 g$$

$$\underline{T = \frac{2 m_1 m_2 g}{\left(m_2 + m_1\right)}}$$

Example

A 3 kg mass and a 5 kg mass are connected over a pulley by a light inextensible string. When the masses are released from rest, what is:
i) the acceleration of each mass?
ii) the tension in the string
(Take g=9.8 ms^{-2} . Answer to 2 d.p.)

i)

$$m_1 = 3\text{ kg} \quad m_2 = 5\text{ kg} \quad g = 9.8\text{ ms}^{-2}$$

for mass m_1, using $f = ma$

$$T - m_1 g = m_1 a$$

$$T - 3g = 3a \qquad (i$$

for mass m_2, using $f = ma$

$$m_2 g - T = m_2 a$$

$$5g - T = 5a \qquad (ii$$

adding equations (i and (ii

$$T - 3g = 3a$$

$$\underline{5g - T = 5a}$$

$$2g = 8a$$

$$a = \frac{g}{4} = \frac{9.8}{4} = 2.45$$

Ans. acceleration of each mass is 2.45 ms^{-2}

ii)

from equation (i above

$$T - 3g = 3a$$

$$T = 3a + 3g$$

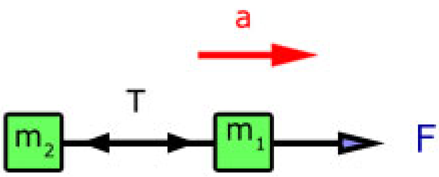
substituting for the values of g and a

$$T = (3\times 2.45) + (3\times 9.8) = 36.75$$

Ans. the tension in the string is 36.75 N

Towe-bar/towe-rope/chains

Usually one body pulled horizontally by another with each linked by a towe-bar or similar. This is similar to the pulley but drawn out in a line.



assuming no friction,

calculation of acceleration 'a'

$$T = m_2 a \qquad (i$$

$$F - T = m_1 a \qquad (ii$$

adding (i and (ii

$$F = (m_1 + m_2)a$$

$$a = \frac{F}{(m_1 + m_2)}$$

calculation of tension 'T'

from (i $\quad T = m_2 a$

substituting for a

$$T = \frac{m_2 F}{(m_1 + m_2)}$$

Example

A car of mass 600 kg towes a trailer of mass 250 kg in a straight line using a rigid towe-bar.
The resistive force on the car is 200N.
The resistive force on the trailer is 80N.
If the forward thrust produced by the engine of the car is 800 N, what is(to 3 d.p.)
i) the acceleration of the car
ii) the tension in the towe-bar

i) looking at all the forces on the car and trailer together

thrust - resistance forces = total mass × acceleration
(tensions forward and backwards cancel out)

$$800 - 80 - 200 = (600 + 250)a$$
$$520 = 850a$$
$$a = \frac{520}{850} = 0.612$$

<u>Ans. acceleration of car is 0.612 ms^{-2}</u>

ii)

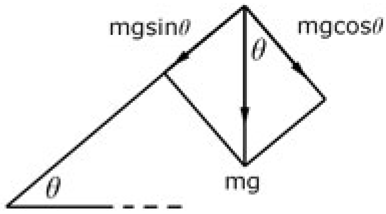
using $f = ma$ for the trailer only

$$T - 80 = 250a$$
$$T = 250a + 80$$
$$T = (250 \times 0.612) + 80 = 233$$

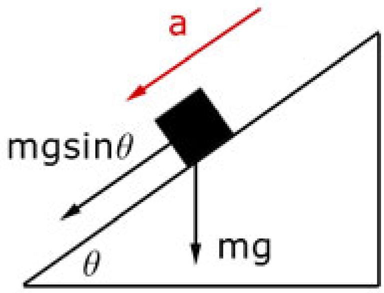
<u>Ans. tension in towe-bar is 233N</u>

Inclined plane with pulley

The pulley at the end just changes the direction of the force. problems involve the resolved component of the weight of the object down the plane.

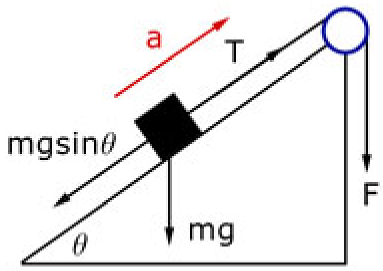


for a mass sliding down a smooth incline



$$mg\sin\theta = ma$$

for a mass pulled up an incline via a pulley



$$F - mg\sin\theta = ma$$

Example

A 2 kg mass on a smooth 30^o plane is connected to a 5 kg mass by a light inextensible string passing over a pulley at the top of the plane.
When the particles are released from rest the 2 kg mass moves up the plane.
i) what is the acceleration of the 2 kg & 5 kg masses?
ii) What is the tension in the string?

i)

T tension in string
m_1 mass being pulled up incline
m_2 mass falling under gravity
a acceleration of both masses
$m_1 = 2$ kg $\quad m_2 = 5$ kg $\quad g = 9.8$ ms^{-2}

consider forces on the 2kg mass, using $f = ma$

$$T - m_1 g \sin 30^\circ = m_1 a$$
$$T - (2 \times 9.8 \times 0.5) = 2a$$
$$T - 9.8 = 2a \qquad \text{(i}$$

consider forces on the kg mass, using $f = ma$

$$m_2 g - T = m_2 a$$
$$(5 \times 9.8) - T = 5a$$
$$49 - T = 5a \qquad \text{(ii}$$

adding equations (i and (ii

$$T - 9.8 = 2a$$
$$\underline{49 - T = 5a}$$
$$49 - 9.8 = 7a$$
$$a = \frac{39.2}{7} = 5.6$$

Ans. acceleration of the 2 kg & 5kg masses is 5.6 ms^{-2}

ii)

substituting into equation (i for a

$$T - 9.8 = 2a$$

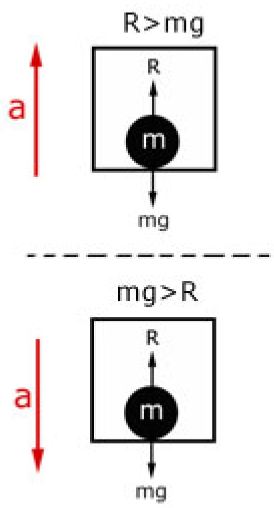
$$T = 2a + 9.8$$

$$= (2 \times 5.6) + 9.8 = 21$$

Ans. tension in string is 21N

Mass ascending or descending in a lift

It is important to remember that there are only two forces on the body in the lift - the weight down and the reaction of the floor up.



<u>Example</u>

A person of mass 100 kg stands in a lift.
What is the force exerted by the lift floor on the person when the lift is:
i) moving upwards at 3 ms^{-1}
ii) moving downwards at 4 ms^{-1}

i)

R reaction upwards of the lift floor
a acceleration of the lift upwards
m mass of person

$m = 100$ kg $\quad a = 3\text{ms}^{-1} \quad g = 9.8 \text{ ms}^{-1}$

considering movement upwards, using $f = ma$

$$\begin{aligned} R - mg &= ma \\ R &= ma + mg \\ &= (100 \times 3) + (100 \times 9.8) \\ &= 300 + 980 = 1280 \end{aligned}$$

<u>Ans. reaction of floor, lift ascending, is 1280 N</u>

ii)

a acceleration of the lift downwards

$a = 4 \text{ ms}^{-1}$

considering movement downwards, using $f = ma$

$$\begin{aligned} mg - R &= ma \\ R &= mg - ma \\ &= (100 \times 9.8) - (100 \times 4) \\ &= 980 - 400 = 580 \end{aligned}$$

<u>Ans. reaction of floor, lift descending, is 580 N</u>

Work & Energy

Theory - Work & Energy

Consider a particle of mass m moving linearly with an applied force F constantly acting on it.
u = initial speed, v = final speed, a = acceleration,
s = distance covered, t = time taken

from Newton's 2nd Law,

$$F = ma \qquad (1$$

from equations describing uniform linear motion

$$v^2 - u^2 = 2as$$

$$\Rightarrow \qquad a = \frac{v^2 - u^2}{2s}$$

substituting for a in equation (1

$$F = m\left(\frac{v^2 - u^2}{2s}\right)$$

$$= \frac{1}{s}\left[\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2\right]$$

$$\underline{Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2}$$

By definition,

work done = (force) x (distance force moves)

Since the expression $\frac{1}{2}mv^2$ is defined as the kinetic energy of a particle of mass m, speed v, our definition is modified to:

work done = change in kinetic energy produced

<u>mathematical proof</u>

Consider a particle, mass m, speed v, being moved along the x-axis by a force of magnitude F.
The applied force F is proportional to the displacement of the particle, x, along the x-axis.

$$F = ma$$

since $\quad a = v\dfrac{dv}{dx}$

$$\Rightarrow \quad F = mv\frac{dv}{dx}$$

$$\Rightarrow \quad Fdx = mvdv$$

$$\Rightarrow \quad \int Fdx = \int mvdv$$

considering the work done when the particle has:
original velocity v_1 at displacement x_1
final velocity v_2 at displacement x_2

$$\Rightarrow \quad \int_{x_1}^{x_2} Fdx = \int_{v_1}^{v_2} mvdv$$

$$F\left[x\right]_{x_1}^{x_2} = \left[\tfrac{1}{2}mv^2\right]_{v_1}^{v_2}$$

$$\underline{F(x_2 - x_1) = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}mv_1^2}$$

Gravitational Potential Energy

This is the energy a mass posesses by virtue of its position. It is equal to the product of mass, gravitational field strength(g) and the vertical distance the particle is above a fixed level.

m = mass speed(ms^{-1}), g = gravitational field strength(N/kg)

h = vertical distance(m)

$$\text{potential energy P.E.(joules)} = mgh$$

(note the unit for g - the gravitational force on a mass of 1kg)

The Law of Conservation of Energy

In a closed system the amount of energy is constant. Or in other words 'energy can never be created nor destroyed', it mearly changes from one form into another.

This is the classical physics view that is useful for most purposes. However, in the real world systems are seldom perfect. We also have the problem when referring to particle physics that energy can indeed be created and destroyed. Annihilation of elementary particles is an example of this(matter-antimatter: electron-positron collision).

Example #1

A pump forces up water at a speed of $8ms^{-1}$ from a well into a reservoir at a rate of 50 kg s^{-1} .
If the water is raised a vertical height of 40 m, what is the work done per second?(assume $g=10\ ms^{-2}$)

$$m = 50\text{kg} \quad h = 40\text{m} \quad g = 10\text{ms}^{-2} \quad v = 8\text{ms}^{-1}$$

$$\begin{aligned}\text{work done} &= \text{increase in PE} + \text{increase in KE}\\ &= mgh + \tfrac{1}{2}mv^2\\ &= (50\times10\times40)+(0.5\times50\times8\times8)\\ &= 20,000+1,600 = 21,600\end{aligned}$$

<u>Ans. work done per second is 21,600J.</u>

Example #2

A gun is fireed at a 3 cm thick solid wooden door.
The bullet, of mass 7g, travels through the door and has speed reduced from 450 ms^{-1} to 175 ms^{-1}.
Assuming uniform resistance, what is the force of the wood on the bullet. (answer to 5 sig. figs.)

$u = 450\text{ms}^{-1} \quad v = 175\text{ms}^{-1} \quad s = 3\text{cm} \equiv 0.03\text{m}$

$m = 7\text{g} \equiv 0.007\text{kg}$

using $\quad v^2 - u^2 = 2as$

$$\begin{aligned} a &= \frac{v^2 - u^2}{2s} \\ &= \frac{(175)^2 - (450)^2}{2 \times 0.03} \\ &= \frac{30625 - 202500}{0.06} \\ &= 3885416.667 \end{aligned}$$

using $\quad F = ma,$

$$\begin{aligned} F &= 0.007 \times 3885416.667 \\ &= 27197.917 \end{aligned}$$

Ans. resistive force from wood is 27198N (5 sig.figs.)

Example #3

In a science experiment, a 50g mass slides down a 60^o incline of length 0.5m.
If the mass is given an initial speed of 2 ms^{-1} down the plane and its final speed is measured as 3 ms^{-1}, what is the magnitude of the frictional force opposing the mass? (assume g=10 ms^{-2} , answer to 2d.p.)

$m = 50\text{g} \equiv 0.05\text{kg} \quad s = 0.5\text{m} \quad u = 2\text{ms}^{-1} \quad v = 3\text{ms}^{-1}$
f frictional force

$$\text{energy at start} = \text{energy at end}$$
$$PE_{\textbf{start}} + KE_{\textbf{start}} = PE_{\textbf{end}} + KE_{\textbf{end}} + \text{work done}$$
$$mgh + \tfrac{1}{2}mu^2 = 0 + \tfrac{1}{2}mv^2 + fS$$
$$(0.05\times10\times0.433) + (0.5\times0.05\times4) = (0.5\times0.05\times9) + 0.5f$$
$$0.2165 + 0.1 = 0.225 + 0.5f$$
$$0.3165 - 0.225 = 0.5f$$
$$f = \frac{0.915}{0.5} = 1.83$$

<u>Ans. frictional force opposing mass movement is 1.83N</u>

Power & Efficiency

Power

Power is by definition the rate of working.
Since **work = force x distance moved**, it follows that :

$$\begin{aligned} \text{power} &= \frac{\text{force} \times \text{distance moved}}{\text{time}} \\ &\equiv \frac{\text{Newtons} \times \text{metres}}{\text{seconds}} \\ &\equiv \text{Nms}^{-1} \end{aligned}$$

1 joule is the work done when 1 newton moves its point of application through 1 metre,

$$1 \text{ joule} = 1 \text{ newton} \times 1 \text{ metre}$$

$$1\text{J} = 1 \text{ Nm}$$

1 watt is a rate of working of 1 joule per second

$$1\text{W} = 1 \text{ Js}^{-1}$$

Example

A military tank of mass 20 metric tonnes moves up a 30^o hill at a uniform speed of 5 ms^{-1} . If all the frictional forces opposing motion total 5000N, what is the power delivered by the engine?
(g = 10ms^{-2} , answer in kW)

If the tank is moving at constant speed then the forces forwards are balanced by the forces backwards.

m is the tank's mass, then mgsin30^o is the component of the weight down the hill
R is the total of resistive forces down the hill
T is the tractive force forwards up the hill

mgsin30^o + R = T

T = (20,000 x 10 x 0.5) + 5000 = 105,000N

power = force x speed

power of tank engine = 105,000 x 5 = 525,000W

Ans. 525 kW

Efficiency

Efficiency is the ratio of useful work out divided by total work done, expressed as a percentage.

$$\text{efficiency}(\%) = \frac{\text{useful work out}}{\text{total work done}} \times 100$$

Example

A pump running at an efficiency of 70% delivers oil at a rate of 4 kgs^{-1} with a speed of 3 ms^{-1} to an oil heater .
if the vertical distance moved by the oil is 10 m, what is the power consumption of the pump?
(g = 10 ms^{-2}, answer to 1 d.p.)

E_f = 70%, m=4kg, v=3 ms^{-1} , h=10 m, g=10 ms^{-2}

work/sec. to raise oil 8 m high = mgh = 4x10x10 = 400 J/s
work/sec. to produce discharge speed = 0.5x4x3x3 = 18 J/s
total work/sec. = 400 + 18 = 418 W

418 W represents 70% of the power supplied

therefore total power consumption of pump =

$$\frac{418}{70} \times \frac{100}{1} = \frac{41800}{70} = 597.143$$

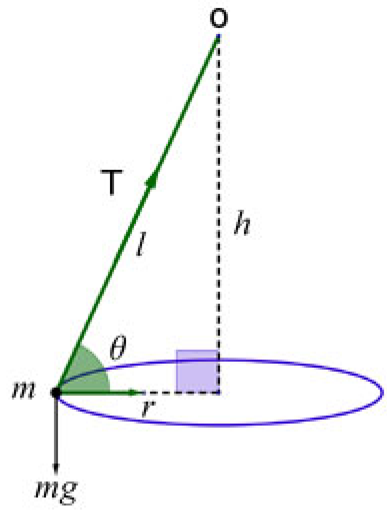
Ans. power consumption of pump is 597.1W (1.d.p.)

Circular Motion

Conical pendulum

Problems concerning the conical pendulum assume no air resistance and that the string has no mass and cannot be stretched.

Solution of problems involves resolving forces on the mass vertically and horizontally. In this way the speed of the mass, the tension in the string and the period of revolution can be ascertained.



resolving forces vertically on the mass

$$mg = T\sin\theta$$

$$T = \frac{mg}{\sin\theta} \qquad \text{(i}$$

resolving forces horizontally on the mass

$$T\cos\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{r\cos\theta} \qquad \text{(ii}$$

eliminating T by combining (i and (ii

$$\frac{mg}{\sin\theta} = \frac{mv^2}{r\cos\theta}$$

$$gr = v^2\tan\theta$$

$$v = \sqrt{\frac{gr}{\tan\theta}}$$

but $\tan\theta = \dfrac{h}{r}$

$\Rightarrow$ $$v = \sqrt{gr\left(\frac{r}{h}\right)}$$

$\Rightarrow$ $$v = \sqrt{\left(\frac{gr^2}{h}\right)}$$

$\Rightarrow$ $$\underline{v = r\sqrt{\left(\frac{g}{h}\right)}}$$

$$\text{the period of revolution} = \frac{\text{circumference}}{\text{speed}}$$

$$= \frac{2\pi r}{v}$$

substituting for v using $v = r\sqrt{\frac{g}{h}}$

$$\text{the period of revolution} = \frac{2\pi r}{r\sqrt{\frac{g}{h}}} = \frac{2\pi r}{r}\sqrt{\frac{h}{g}}$$

$$\underline{\text{the period of revolution} = 2\pi\sqrt{\frac{h}{g}}}$$

Example

A 20g mass moves as a conical pendulum with string length 8x and speed v.
if the radius of the circular motion is 5x find:
i) the string tension(assume g =10 ms^{-2} , ans. to 2 d.p.)
ii) v in terms of x, g

i)

$$l = 8x \qquad r = 5x \qquad \cos^{-1}\theta = \tfrac{5}{8} \qquad \theta = 51.3^{\circ}$$

$$m = 20\text{g} \equiv 0.02\text{kg} \qquad g = 10\text{ms}^{-2}$$

resolving vertically

$$T\sin\theta = mg$$

$$T = \frac{mg}{\sin\theta}$$

$$= \frac{0.02 \times 10}{0.7804} = 0.2563$$

$\underline{\text{Ans. the string tension } T \text{ is } 0.26\text{N (2.d.p.)}}$

ii)

resolving horizontally

$$T\cos\theta = \frac{mv^2}{r}$$

substituting for T, from $T = \dfrac{mg}{\sin\theta}$ above

$$mg\frac{\cos\theta}{\sin\theta} = \frac{mv^2}{r}$$

$$\frac{gr}{\tan\theta} = v^2$$

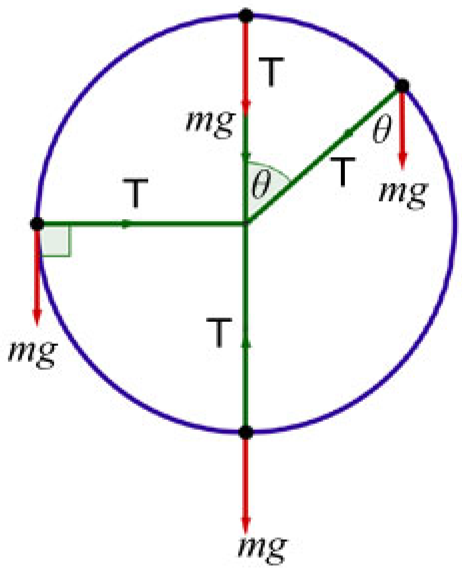
$$v = \sqrt{\frac{gr}{\tan\theta}}$$

substituting for $r = 5x$, $\theta = 51.3^\circ$

$$v = \sqrt{\frac{5gx}{\tan(51.3^\circ)}} = \sqrt{\frac{5}{\tan(51.3^\circ)}}.\sqrt{gx} = 2\sqrt{gx}$$

<u>Ans. velocity v in terms of g, x is $2\sqrt{gx}$</u>

<u>Mass performing vertical circular motion under gravity</u>



Consider a mass m performing circular motion under gravity, the circle with radius r .
The centripetal force on the mass varies at different positions on the circle.

top $\quad mg + T = \dfrac{mv^2}{r}$

middle $\quad T = \dfrac{mv^2}{r}$

bottom $\quad T - mg = \dfrac{mv^2}{r}$

string at an angle θ to the vertical

$$mg\cos\theta + T = \frac{mv^2}{r}$$

For many problems concerning vertical circular motion, energy considerations(KE & PE) of particles at different positions are used to form a solution.

Example #1

A 50g mass suspended at the end of a light inextensible string performs vertical motion of radius 2m.
if the mass has a speed of 5 ms^{-1} when the string makes an angle of 30^{o} with the vertical, what is the tension?
(assume g =10 ms^{-2} , answer to 1 d.p.)

$m = 50\text{g} \equiv 0.05\text{kg} \quad v = 5\text{ms}^{-1} \quad \theta = 30^\circ \quad r = 2\text{m}$

$g = 10\text{ms}^{-2}$

the centripetal force is the sum of the tension in the string and the component of the weight along the string

$$\Rightarrow \quad mg\cos\theta + T = \frac{mv^2}{r}$$

$$\Rightarrow \quad T = \frac{mv^2}{r} - mg\cos\theta$$

$$= \frac{(0.05)(5)^2}{2} - (0.05)(10)\cos 30^\circ$$

$$= 0.625 - 0.433 = 0.192$$

<u>Ans. tension in string is 0.2N</u>

Example #2

A 5kg mass performs circular motion at the end of a light inextensible string of length 3m. If the speed of the mass is 2 ms^{-1} when the string is horizontal, what is its speed at the bottom of the circle?
(assume g =10 ms^{-2})

$v_H = 2\text{ms}^{-1}$ $\quad r = 3\text{m}$ $\quad g = 10\text{ms}^{-1}$

v_B speed at bottom of circle

PE is measured relative to the bottom of the circle

KE + PE string horizontal = KE + PE at bottom

$$\tfrac{1}{2}mv_H^2 + mgr = \tfrac{1}{2}mv_B^2 + 0$$

$$v_H^2 + 2gr = v_B^2$$

$$v_B = \sqrt{v_H^2 + 2gr}$$

$$= \sqrt{(2)^2 + 2\times(10)\times(3)}$$

$$= \sqrt{4+60} = \sqrt{64} = 8$$

Ans. speed at bottom of circle is 8 ms^{-1}

Elastic Strings / Springs

Hooke's Law

The extension e of an elastic string/spring is proportional to the extending force F.

$$F \propto e$$
$$F = ke$$

where k is the constant of proportionality, called the spring constant.

By definition the **spring constant** is that force which will produce unit extension(unit Nm^{-1}) in a spring.

example

A 2kg mass hangs at the end of an elastic string of original length 0.5 m.
If the spring constant is 60 Nm^{-1}, what is the length of the extended elastic string?

$$m = 2\text{kg} \qquad k = 60\text{Nkg}^{-1} \quad l_o = 0.5\text{m}$$
$$e = l - l_o \qquad g = 10\text{ms}^{-2}$$

resolving forces vertically on mass

$$mg = ke$$

substituting for $e = l - l_o$

$$mg = k(l - l_o)$$
$$\frac{mg}{k} = (l - l_o)$$
$$l = l_o + \frac{mg}{k}$$
$$l = 0.5 + \left(\frac{2 \times 10}{60}\right) = 0.8\dot{3}$$

Ans. length of extended string is 0.8 m(1 d.p.)

The energy stored in a stretched string/spring
(same as the work done in stretching a string)

Remembering that 'work = force x distance moved', if the elastic string is extended a small amount δe(such that the force F is considered constant) then the work done δW is given by:

$$\delta W = F\delta e \qquad (i$$

$$\text{substituting for} \quad F = ke \quad \text{in} \quad (i$$

$$\Rightarrow \qquad \delta W = ke\delta e$$

$$\text{work } W \text{ done in extension } 0 \text{ to } e$$

$$W = \int_0^e ke\,de$$

$$= k\int_0^e e\,de$$

$$= k\left[\frac{e^2}{2}\right]_0^e$$

$$= \frac{ke^2}{2}$$

$$\underline{W = \tfrac{1}{2}ke^2}$$

example

A 2 kg mass m hangs at a point B at the end of an elastic string of natural length 0.7 m supported at a point A.
The extension produced in the string by the mass is a fifth of the original length, while the spring constant is $5mg$.
If the mass is now held at point A and then released, what is the maximum speed the mass will attain?
(g = 10 ms^{-2} , answer to 1 d.p.)

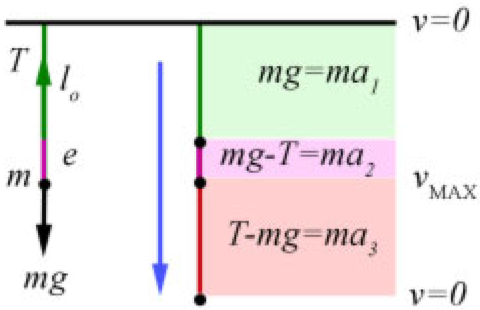
Let PE be related to the equilibrium position of the mass. Hence the mass will have maximum KE when passing through point B. Below that point there will be no net force downwards to produce acceleration. The net force will act in the opposite direction, producing deceleration.

In the diagram,

a_1 is linear acceleration under gravity

a_2 is non-linear acceleration(decreasing)

a_3 is negative acceleration i.e. deceleration.



$$m = 2\text{kg} \quad l_o = 0.7\text{m} \quad e = \frac{l_o}{5} \quad l = \frac{6}{5}\, l_o$$

$$g = 10\text{ms}^{-2} \quad k = 5mg$$

$$\text{PE at A} = (\text{KE at B}) + (\text{elastic PE at B})$$

$$mgl = \frac{1}{2}mv^2 + \frac{1}{2}ke^2$$

$$\frac{6}{5}mgl_o = \frac{1}{2}mv^2 + \frac{5}{2}mg\left(\frac{l_o}{5}\right)^2$$

$$\frac{6}{5}gl_o = \frac{v^2}{2} + \frac{gl_o^{\,2}}{10}$$

multiplying both sides by 10

$$12gl_o = 5v^2 + gl_o^{\,2}$$

$$5v^2 = 12gl_o + gl_o^{\,2}$$

$$v = \sqrt{\frac{12gl_o + gl_o^{\,2}}{5}}$$

$$= \sqrt{\frac{(12 \times 10 \times 0.7) + (10 \times 0.7 \times 0.7)}{5}}$$

$$= 4.2166$$

<u>Ans. maximum speed of mass is 4.2ms^{-1} (1d.p.)</u>

Notes