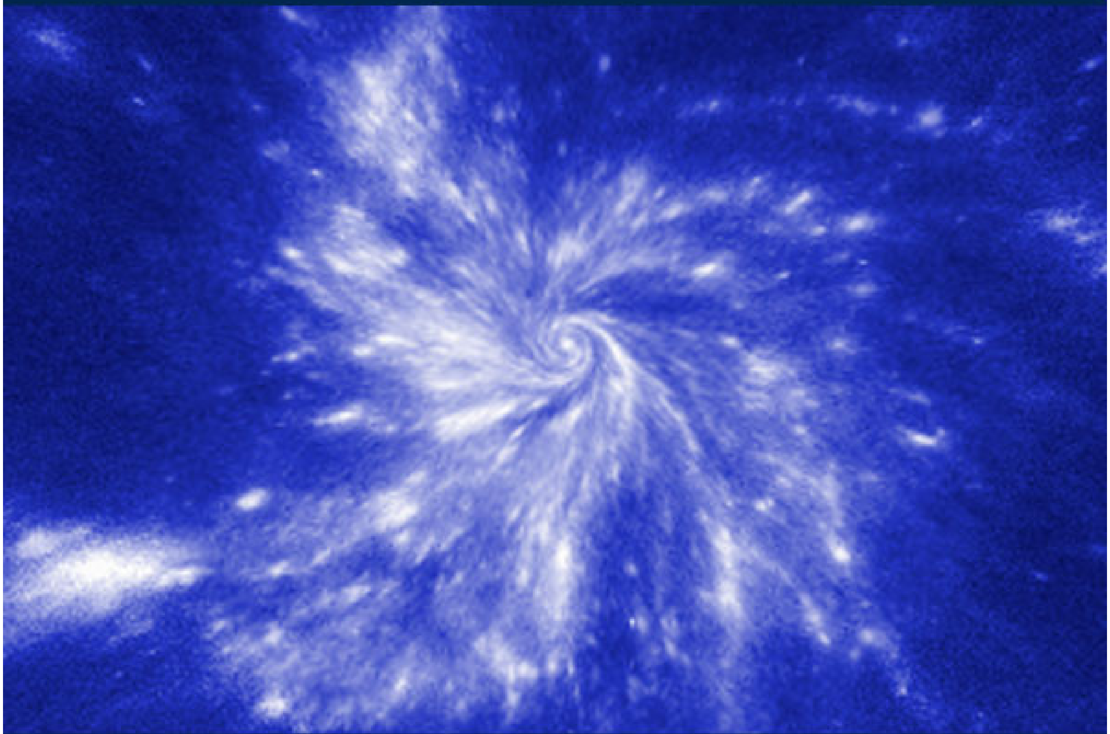
A-LEVEL MATHS TUTOR

Pure Maths



PART ONE

DIFFERENTIAL CALCULUS

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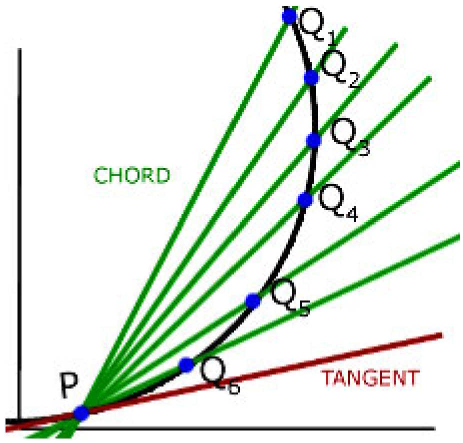
The Derivative Formula

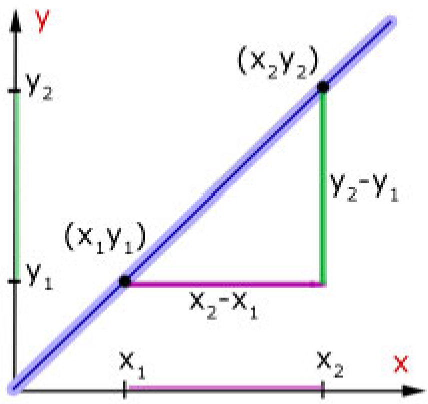
First Principles



To find an expression for the gradient of the tangent at point P on a curve, we must consider lines passing through P and cutting the curve at points Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 ...etc.

As Q approaches P so the gradient of the chord PQ approaches the gradient of the tangent at P.

We can form an expression for the gradient at P by using this concept.

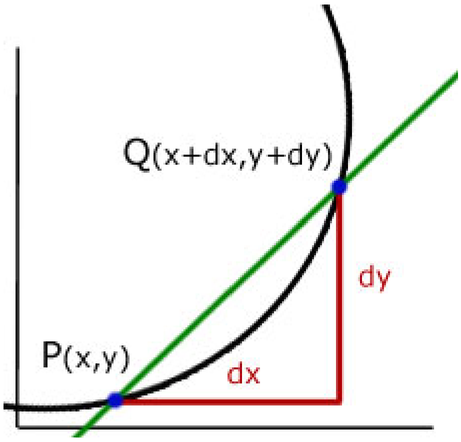


We know from coordinate geometry that:

$$\text{gradient } = \frac{y_2 - y_1}{x_2 - x_1}$$

for points (x_1, y_1) and (x_2, y_2)

Consider the coordinates of P to be *(x,y)* and point Q to be *(x+dx, y+dy)*, where *dx* and *dy* are the horizontal and vertical components of the line PQ.



Gradient of the line between points *(x,y)* and *(x+dx, y+dy)* is given by :

$$\text{gradient} = \frac{(y+dy)-y}{(x+dx)-x} = \frac{y+dy-y}{x+dx-x} = \frac{dy}{dx}$$

The tangent to the curve = gradient of PQ when the length of PQ is zero and $dx = 0$ and $dy = 0$.

in the limit, as *dx* '<u>approaches zero</u>' the gradient of the curve is said to be *dy/dx*.

If we now replace ***y*** by ***f(x)*** in the expression for gradient, since ***y = f(x)*** i.e. *y* is a function of *x*.

$$\lim_{dx\to 0} \frac{(y+dy-y)}{(x+dx-x)}$$

and

$$y = f(x)$$

$$y + dy = f(x + dx)$$

we have:

$$\lim_{dx\to 0} \frac{f(x+dx)-f(x)}{dx}$$

that is,

$$\frac{dy}{dx} = \lim_{dx\to 0} \frac{f(x+dx)-f(x)}{dx}$$

<u>Example</u>: find the gradient of $y = 4x^2$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4(x+dx)^2 - 4(x)^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4(x^2 + 2xdx + (dx)^2) - 4x^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{4x^2 + 8xdx + 4(dx)^2 - 4x^2}{dx}$$

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{8xdx + 4(dx)^2}{dx}$$

cancelling by dx

$$\frac{dy}{dx} = 8x + 4dx$$

in the limit when $dx = 0$ this becomes,

$$\frac{dy}{dx} = 8x$$

Without doubt this is a very long winded way to work out gradients. There is a simpler way, by using the Derivation Formula(see further down the page).

Notation This is best described with an example.

If $y = 3x^2$, which can also be expressed as $f(x) = 3x^2$, then

the derivative of y with respect to x can be expressed as:

$$\frac{dy}{dx} = 6x \qquad \frac{d(3x^2)}{dx} = 6x \qquad f'(x) = 6x$$

The Derivation Formula

If we have a function of the type $y = k\,x^n$, where $\mathbf{k}$ is a

constant, then,

$$\frac{d(\mathrm{k}\,x^{\mathrm{n}})}{dx} = \mathrm{k\,n}\,x^{\mathrm{n}-1}$$

Example:

Find the gradient to the curve $y = 5\,X^2$ at the point (2,1).

gradient = (5) (2 X^{2-1}) = 10 X^1 = 10 X

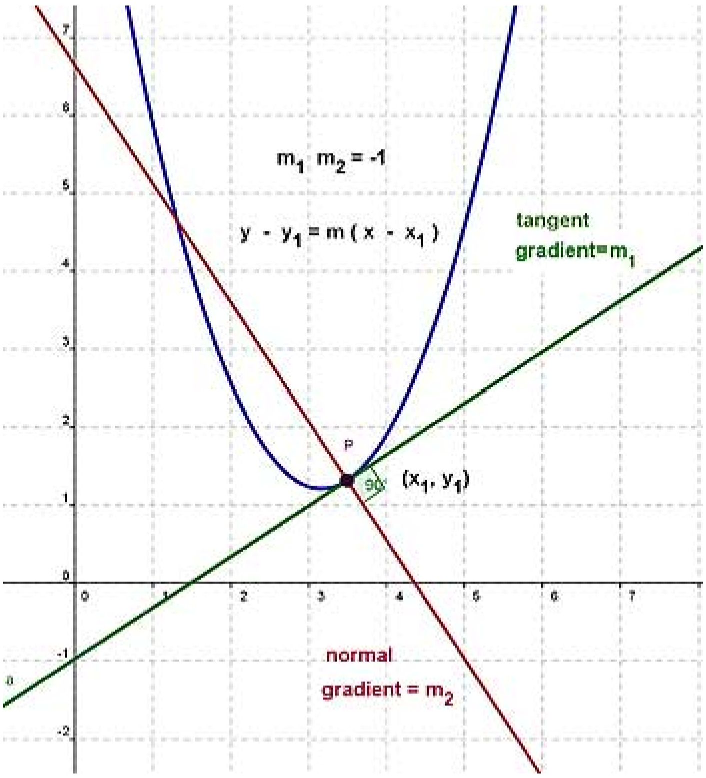
gradient at point (2,1) is 10 x 2 = **20**

Tangents & Normals

Tangents

The gradient of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) on the curve is given by:

the value of dy/dx, when $x = x_1$ and $y = y_1$



Normals

Two lines of gradients m_1, m_2 respectively are perpendicular to eachother if the product,

$$m_1 \times m_2 = -1$$

Equation of a tangent

The equation of a tangent is found using the equation for a straight line of gradient **m**, passing through the point (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

To obtain the equation we substitute in the values for x_1 and y_1 and **m** (dy/dx) and rearrange to make y the subject.

Example

Find the equation of the tangent to the curve $y = 2x^2$ at the point (1,2).

$$y = 2x^2$$

therefore gradient, $\dfrac{dy}{dx} = 4x$

when $x = 1$ $\quad \dfrac{dy}{dx} = 4$ $\quad$ therefore gradient is 4

using $\quad y - y_1 = m(x - x_1)$ *

*from coordinate geometry, gradient m of the line between two points (x, y) and (x_1, y_1)

$x_1 = 1 \qquad y_1 = 2$

$$y - 2 = 4(x - 1)$$
$$y - 2 = 4x - 4$$
$$y = 4x - 4 + 2$$
$$\underline{y = 4x - 2}$$

Equation of a normal

The equation of a normal is found in the same way as the tangent. The gradient(**m_2**)of the normal is calculated from;

m_1 x **m_2 = -1** (where **m_1** is the gradient of the tangent)

so

$$m_2 = -1/(m_1)$$

Example

Find the equation of the normal to the curve:

$y = x^2 + 4x + 3$, at the point (-1,0).

$$y = x^2 + 4x + 3$$

therefore gradient(m_1), $\dfrac{dy}{dx} = 2x + 4$

at the point (-1,0) $m_1 = 2x + 4 = -2 + 4 = 2$

let the gradient of the normal be m_2

product of tangent and normal gradient:

$$m_1.m_2 = -1$$

$m_1 = 2$

$\therefore \quad 2.m_2 = -1 \qquad m_2 = -\frac{1}{2}$

using $\quad y - y_1 = m_2(x - x_1)$

when $\quad x_1 = -1, \qquad y_1 = 0$

$$y - 0 = -\tfrac{1}{2}(x - (-1))$$

$$y = -\tfrac{1}{2}x - \tfrac{1}{2}$$

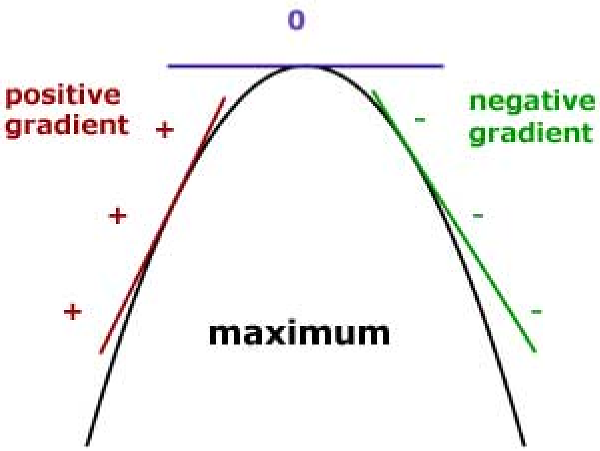
mult. by 2 $\quad 2y = -x - 1$

$$\underline{2y + x + 1 = 0}$$

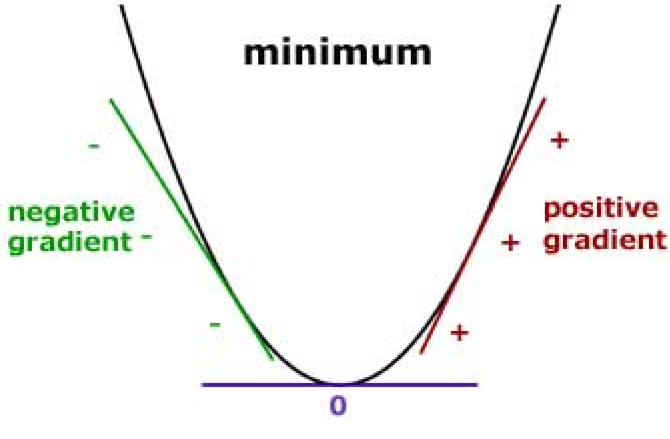
Maxima & Minima

Gradient change

Starting to the **left** of a **maximum** the gradient changes from ' **+** ' to ' **-** 'with increasing 'x'.



Starting to the **left** of a **minimum**, the gradient changes from ' **-** ' to ' **+** 'with increasing 'x'.



At the point of maximum or minimum the gradient is zero.

Example

Show that the curve $y = x^2$ has a minimum at (0,0).

$$y = x^2$$

gradient, $\dfrac{dy}{dx} = 2x$

at the point (0,0) $\quad x = 0$

gradient, $\dfrac{dy}{dx} = 2x = 0$

$\therefore$ there is either a maximum or minimum at (0,0)

taking a value of x less than 0, say -1

gradient, $\dfrac{dy}{dx} = 2x = -2$ (a negative gradient)

taking a value of x more than 0, say +1

gradient, $\dfrac{dy}{dx} = 2x = 2$ (a positive gradient)

The gradient changes from negative to positive with increasing x,

$\therefore$ the function has a minimum at (0,0)

Locating the point of maximum or minimum

The x-value at a maximum or minimum is found by differentiating the function and putting it equal to zero.

The y-value is then found by substituting the 'x' into the original equation.

Example

Find the coordinates of the greatest or least value of the function:

$$y = x^2 + 3x + 2$$

$$\text{gradient, } \frac{dy}{dx} = 2x + 3$$

max. or min. when gradient is zero

$$0 = 2x + 3$$

$$x = -\tfrac{3}{2} = \underline{-1.5}$$

substituting this value of x into $y = x^2 + 3x + 2$

$$y = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2$$

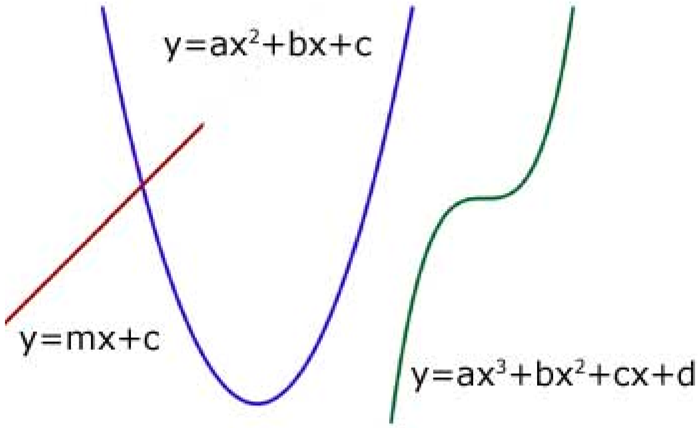
$$y = \left(\frac{9}{4}\right) - \left(\frac{9}{2}\right) + 2$$

$$= 2.25 - 4.5 + 2 = \underline{-0.25}$$

coords. of the maximum/minimum are (-1.5,-0.25)

Curve Sketching

The power of 'x' gives a hint to the general shape of a curve.



Together with the point of maximum or minimum, where the curve crosses the axes at y=0 and x=0 gives further points.

<u>Example</u>

Sketch the curve $y = x^2 + 3x + 2$ from the example above, given that there is a minimum at (-1.5,-0.25).

factorising and putting y=0 to find where the curve crosses the x-axis,

$$(x+1)(x+2)=0$$

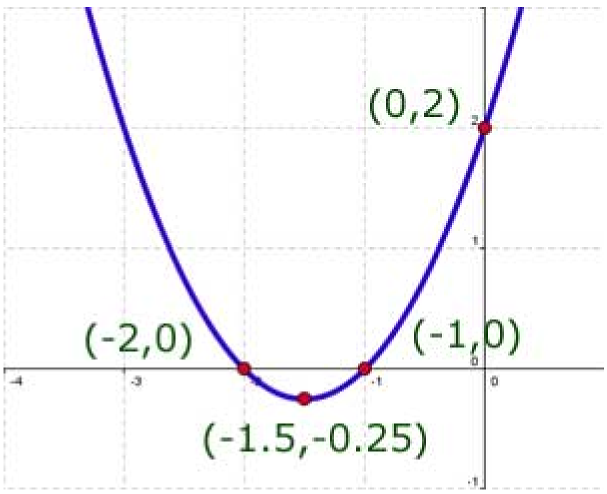
$$x=-1 \text{ and } x=-2$$

so the curve crosses the x-axis at (-1,0) and (-2,0)

putting x=0 to find where the curve crosses the y-axis

$$y=2$$

so the curve crosses the y-axis at (0,2)



The Chain Rule

The Chain Rule Equation

This is a way of differentiating a function of a function.

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

Example #1

differentiate $(3x+3)^3$

let $y = (3x+3)^3$ and $t = 3x+3$

then $y = t^3$

$\frac{dt}{dx} = 3$, $\qquad \frac{dy}{dt} = 3t^2$

using the Chain Rule

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$, $\qquad \therefore \frac{dy}{dx} = 3t^2.3 = 9t^2$

$$\frac{d\left\{(3x+3)^3\right\}}{dx} = 9(3x+3)^2 = 9(3)(x+1)(3)(x+1)$$

$$\underline{= 81(x+1)^2}$$

Example #2

differentiate $(x^2+5x)^6$

let $y = (x^2+5x)^6$ and $t = x^2+5x$

then $y = t^6$

$\frac{dt}{dx} = 2x+5$, $\qquad \frac{dy}{dt} = 6t^5$

using the Chain Rule

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$, $\qquad \frac{dy}{dx} = 6t^5.2x+5$

$$\frac{d\left\{(x^2+5x)^6\right\}}{dx} = 6(x^2+5x)^5(2x+5)$$

$$\underline{= 6(2x+5)(x^2+5x)^5}$$

Rates of change

The Chain Rule is a means of connecting the rates of change of dependent variables.

Example #1

If air is blown into a spherical balloon at the rate of 10 cm^3 how quickly will the radius grow?

if the radius of the balloon is r

then the volume $V = \frac{4}{3}\pi r^3$

and $\frac{dV}{dr} = 4\pi r^2$

the rate of change of volume with time

is given by: $\frac{dV}{dt} = 10\text{cm}^3 / \text{sec.}$

using the Chain Rule

$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$ and $\frac{dV}{dr} = 4\pi r^2$

$$\therefore \frac{dr}{dt} = \frac{dV}{dt}.\frac{1}{\frac{dV}{dr}} = \frac{dV}{dt}.\frac{dr}{dV} = 10.\frac{1}{4\pi r^2}$$

$$= \frac{5}{2\pi r^2}$$

i.e. rate of change of radius is $\frac{5}{2\pi r^2}$ cm/sec.

Example #2

A spherical raindrop is formed by condensation. In an interval of 10 sec. its volume increases at a constant rate from 0.010mm^3 to 0.500mm^3.

Find the rate at which the surface area of the raindrop is increasing, when its radius is 1.0mm

radius r mm

volume V is given by: $V = \frac{4}{3}\pi r^3$

$$\therefore \quad \frac{dV}{dr} = \frac{4\pi}{3}.3r^2 = 4\pi r^2$$

also, area A is given by: $A = 4\pi r^2$

$$\therefore \quad \frac{dA}{dr} = 4\pi.2r = 8\pi r$$

vol. increases at a constant rate by:

$0.5 - 0.010 = 0.490 \text{ mm}^3$ in 10 sec.

so $\frac{dV}{dt} = \frac{0.49}{10} = 0.049 \text{ mm}^3.s^{-1}$.

we are required to find $\frac{dA}{dt}$ when $r = 1.0\text{mm}$

using the Chain Rule,
$$\frac{dA}{dt} = \left(\frac{dA}{dV}\right).\frac{dV}{dt}$$
$$= \left(\frac{dA}{dr}.\frac{dr}{dV}\right).\frac{dV}{dt}$$

$$\frac{dA}{dt} = \left(8\pi r.\frac{1}{4\pi r^2}\right)0.049 = \left(\frac{2\times 0.049}{r}\right) = \frac{0.098}{r}$$

when $r = 1.0\text{mm}$, $\frac{dA}{dt} = \frac{0.098}{1} = 0.098\text{mm}^2.s^{-1}$.

$\therefore$ surface area, for a radius of 1mm, increases by $0.098\text{mm}^2.s^{-1}$.

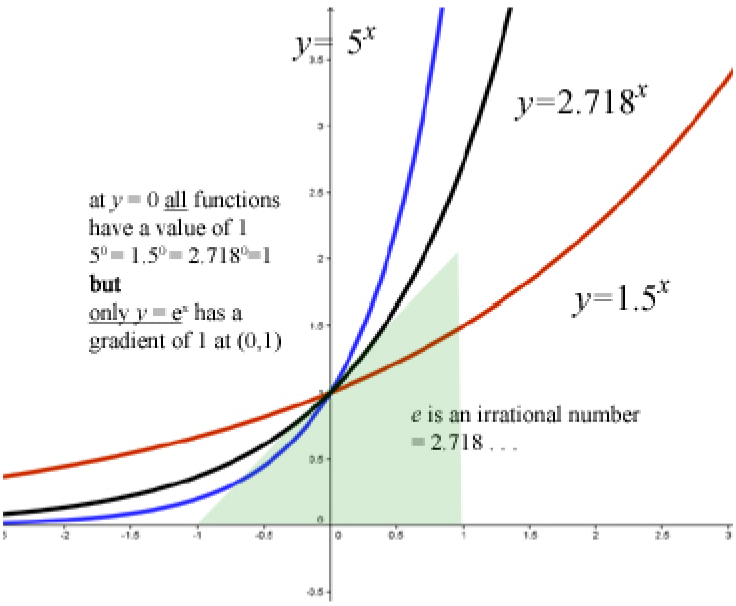
Exponentials & Logarithms

Exponential functions

Strictly speaking **all** functions where the variable is in the index are called exponentials.

The Exponential function e^x

This is the **one** particular exponential function where 'e' is approximately 2.71828 and the gradient of $y = e^x$ at (0,1) is 1.



One other special quality of $y = e^x$ is that its derivative is also equal to e^x

$$\frac{d(e^x)}{dx} = e^x$$

and for problems of the type y= e^{kx}

$$t = kx \qquad \frac{dt}{dx} = k$$

$$y = e^t \qquad \frac{dy}{dt} = e^t = e^{kx}$$

$$\frac{d\left(e^{kx}\right)}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = e^{kx}.k$$

$$\underline{\frac{dy}{dx} = ke^{kx}}$$

Derivative problems like the above concerning 'e' are commonly solved using the Chain Rule.

<u>Example #1</u>

Find the derivative of:

$$y = e^{2x^3}$$

$$\textbf{let } t = 2x^3, \qquad y = e^t$$

$$\frac{dx}{dt} = 6x^2 \qquad \frac{dy}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dx}{dt} = e^t.6x^2$$

$$\underline{\frac{dy}{dx} = e^{2x^3}\,6x^2}$$

Example #2

find the derivative of:

$$y = e^{(3x-4)^2}$$

$$\textbf{let} \quad t = (3x-4)^2 \qquad y = e^t$$

$$\frac{dt}{dx} = 2(3x-4).3 \ , \qquad \frac{dy}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = e^t.6(3x-4)$$

$$\frac{dy}{dx} = \underline{6(3x-4)e^{(3x-4)^2}}$$

Derivative of a Natural Logarithm function

Remember $\mathbf{y=log_e x}$ means:

x is the number produced when **e** is raised to the power of **y**

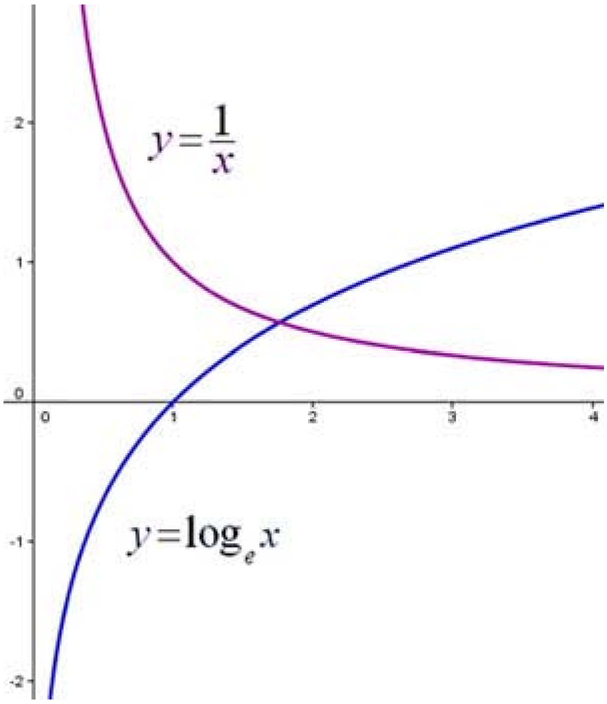
The connection between $y=e^x$ and $y=\log_e x$ can be shown by rearranging $y=\log_e x$.

$\mathbf{y=log_e x}$ can be written as $\mathbf{x=e^y}$

($\mathbf{log_e x}$ is now more commonly written as **ln(x)**)

The derivative of ln(x) is given by:

$$\frac{d(\ln(x))}{dx} = \frac{1}{x}$$



Example #1

find the derivative of y = ln(3x)

$$y = \ln(3x)$$
$$= \ln(3) + \ln(x)$$
$$\frac{dy}{dx} = 0 + \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{1}{x}$$

Example #2

find the derivative of y = ln(x^3+3)

$$y = \ln(x^2 + 3)$$
$$\text{let } \; t = x^2 + 3, \quad \text{then } \; y = \ln(t)$$
$$\frac{dt}{dx} = 2x \; , \qquad \frac{dy}{dt} = \frac{1}{t}$$
$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{t}.2x = \frac{1}{x^2+3}.2x$$
$$\underline{\frac{dy}{dx} = \frac{2x}{x^2+3}}$$

Problems of the type y=$N^{f(x)}$

Problems of this type are solved by taking logs on both sides and/or using the Chain Rule.

Example #1

find the derivative of y=10^x

$$y = 10^x$$
$$\ln(y) = \ln(10^x)$$
$$\ln(y) = x\ln(10)$$
$$\frac{d(\ln(y))}{dx} = \frac{d(x\ln(10))}{dx}$$
$$\frac{d(\ln(y))}{dy}.\frac{dy}{dx} = 1.\ln(10)$$
$$\frac{1}{y}.\frac{dy}{dx} = \ln(10)$$
$$\frac{dy}{dx} = y\ln(10) = \underline{10^x\ln(10)}$$

Example #2

find the derivative of y= ln($\cos^3 2x$)

$$y = \ln(\cos^3 2x)$$

$$t = \cos(2x) \qquad y = \ln(t^3)$$

$$\frac{dt}{dx} = -2\sin(2x) \qquad \frac{dy}{dt} = \frac{1}{t^3}.3t^2 = \frac{3}{t}$$

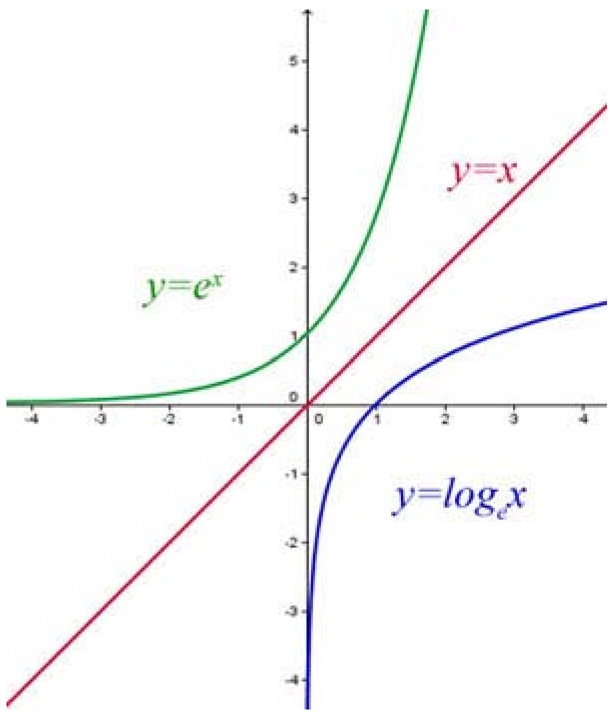
$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{3}{t}.(-2\sin(2x))$$

$$= \frac{3}{\cos(2x)}.(-2\sin(2x))$$

$$= \underline{-6\tan(2x)}$$

A graphical comparison of exponential and log functions

As you can see, y= e^x is reflected in the line y=x to produce the curve y=ln(x)



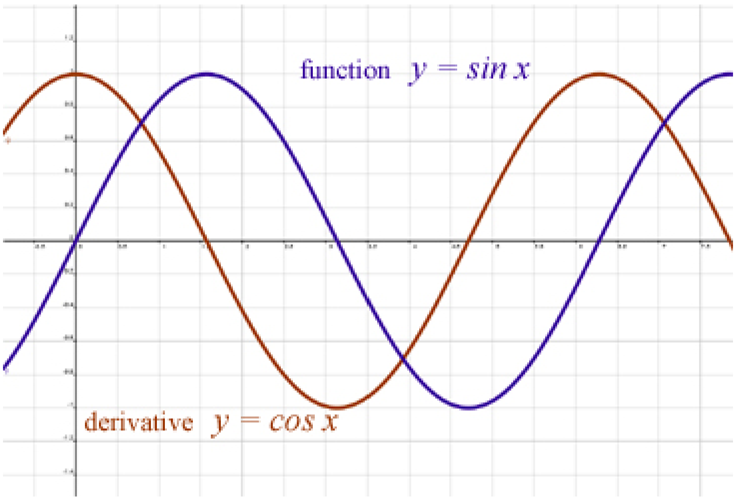
Derivation of Trigonmetrical Functions

Relation between derived trigonometrical functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \operatorname{cotan} x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

Derivative of the Sine Function



$$\frac{d(\sin x)}{dx} = \cos x$$

Example differentiate sin(2x+4)

$$\text{let} \quad y = \sin(2x+4) \quad \text{and} \quad t = 2x+4$$

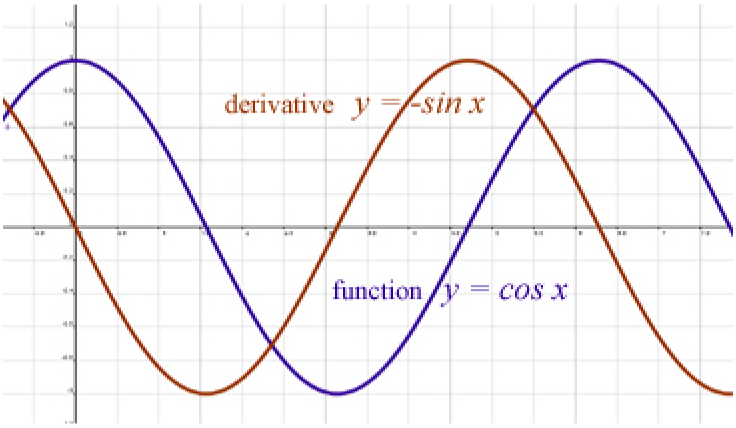
$$\therefore \quad y = \sin(t)$$

$$\frac{dy}{dt} = \cos(t) \qquad \frac{dt}{dx} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \cos(t).2$$

$$= \underline{2\cos(2x+4)}$$

Derivative of the Cosine Function



$$\frac{d(\cos x)}{dx} = -\sin x$$

Example differentiate $\cos^3 x$

$$\text{let} \quad y = \cos^3 x \quad \text{and} \quad t = \cos x$$

$$\therefore \quad y = t^3$$

$$\frac{dy}{dt} = 3t^2 \qquad\qquad \frac{dt}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = 3t^2.(-\sin x)$$

$$= -3\cos^2 x \sin x$$

$$= -\frac{3}{2}\cos x(2\cos x \sin x)$$

$$\text{but} \quad \sin 2\theta = 2\cos\theta \sin\theta$$

$$\underline{\therefore \frac{dy}{dx} = -\frac{3}{2}\cos x \sin 2x}$$

Derivative of the Tangent Function

$$\frac{d(\tan x)}{dx} = \frac{d\left(\dfrac{\sin x}{\cos x}\right)}{dx}$$

using the Product Rule

$$y = \frac{u}{v} \qquad u = \sin x \quad v = \cos x$$

$$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(\cos x).(\cos x) - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\underline{\frac{d(\tan x)}{dx} = \sec^2 x}$$

Derivative of the Cosecant Function

$$\frac{d(\operatorname{cosec} x)}{dx} = \frac{d\left(\dfrac{1}{\sin x}\right)}{dx} = \frac{d(\sin x)^{-1}}{dx}$$

let $\quad y = (\sin x)^{-1}, \quad t = \sin x$

then $\quad y = t^{-1}$

$$\frac{dy}{dt} = (-1)(t^{-2}) \qquad \frac{dt}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -t^{-2}.\cos x = -(\sin x)^{-2}.\cos x$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x.\sin x} = -\frac{\cos x}{\sin x}.\frac{1}{\sin x}$$

$$\underline{\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cotan} x.\operatorname{cosec} x}$$

Derivative of the Secant Function

$$\frac{d(\sec x)}{dx} = \frac{d\left(\dfrac{1}{\cos x}\right)}{dx} = \frac{d(\cos x)^{-1}}{dx}$$

let $\quad y = (\cos x)^{-1}\ , \quad t = \cos x$

then $\quad y = t^{-1}$

$$\frac{dy}{dt} = (-1)(t^{-2}) \qquad \frac{dt}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -t^{-2}.(-\sin x) = (\cos x)^{-2}.\sin x$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x.\cos x} = \frac{\sin x}{\cos x}.\frac{1}{\cos x}$$

$$\underline{\frac{d(\sec x)}{dx} = \tan x.\sec x}$$

Derivative of the Cotangent Function

$$\frac{d(\cot x)}{dx} = \frac{d\left(\dfrac{\cos x}{\sin x}\right)}{dx}$$

using the Product Rule

$$y = \frac{u}{v} \qquad u = \cos x \quad v = \sin x$$

$$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(\sin x).(-\sin x) - (\cos x).(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\mathrm{cosec}^2 x$$

$$\underline{\frac{d(\cot x)}{dx} = -\mathrm{cosec}^2 x}$$

The Product Rule

The Product Rule Equation

Gottfried Leibniz is credited with the discovery of this rule which he called **Leibniz's Law**.

Simply, if **u** and **v** are two differentiable functions of *x*, then the differential of ***uv*** is given by:

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

this can also be written, using '**prime notation**' as :

$$(u.v)' = u.v' + vu'$$

Example #1

$$\text{differentiate} \quad (x^2+1)^3(x^3+1)^2$$

$$u=(x^2+1)^3 \qquad v=(x^3+1)^2$$

$$y=u.v$$

$$\frac{du}{dx}=3(x^2+1)^2.2x \qquad \frac{dv}{dx}=2(x^3+1).3x^2$$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{dy}{dx}=(x^2+1)^3.2(x^3+1).3x^2+(x^3+1)^2.3(x^2+1)^2.2x$$

$$=6x^2(x^2+1)^3.(x^3+1)+6x(x^2+1)^2(x^3+1)^2$$

$$=6x(x^2+1)^2(x^3+1)\{x(x^2+1)+(x^3+1)\}$$

$$=6x(x^2+1)^2(x^3+1)\{x^3+x+x^3+1\}$$

$$=\underline{6x(x^2+1)^2(x^3+1)(2x^3+x+1)}$$

Example #2

$$\text{differentiate} \quad y=(x^2-4)(x+3)^2$$

$$u=(x^2-4) \qquad v=(x+3)^2$$

$$y=u.v$$

$$\frac{du}{dx}=2x \qquad \frac{dv}{dx}=2(x+3)$$

$$\text{using} \quad \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{dy}{dx}=(x^2-4).2(x+3)+(x+3)^2.2x$$

$$=2(x+3)(x^2-4)+2x(x+3)^2$$

$$=2(x+3)\{x^2-4+x^2+3x\}$$

$$=\underline{2(x+3)(2x^2+3x-4)}$$

Example #3

$$\text{differentiate} \qquad y = (x^2+3)\sqrt{(2+x)}$$

$$u = (x^2+3) \qquad v = (2+x)^{1/2}$$

$$y = u.v$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{1}{2}(2+x)^{-1/2}$$

$$\text{using} \qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2+3).\frac{1}{2}(2+x)^{-1/2} + (2+x)^{1/2}.2x$$

$$= \frac{(x^2+3)}{2(2+x)^{1/2}} + \frac{2x(2+x)^{1/2}}{1}$$

$$= \frac{(x^2+3) + 2(2+x)^{1/2}.2x(2+x)^{1/2}}{2(2+x)^{1/2}}$$

$$= \frac{(x^2+3) + 4x(2+x)}{2(2+x)^{1/2}}$$

$$= \frac{(x^2+3) + 8x + 4x^2}{2(2+x)^{1/2}}$$

$$= \frac{5x^2+8x+3}{2(2+x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{(5x+3)(x+1)}{2(2+x)^{1/2}}$$

The Quotient Rule

The Quotient Rule Equation

This is a variation on the Product Rule(**Leibniz's Law**) from the previous topic.

As with the Product Rule, , if **u** and **v** are two differentiable functions of *x*, then the differential of ***u/v*** is given by:

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

this can also be written, using '**prime notation**' as :

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

Example #1

$$\text{differentiate} \quad \frac{(x-3)^2}{(x+2)^2}$$

$$u = (x-3)^2 \qquad v = (x+2)^2$$

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = 2(x-3) \qquad \frac{dv}{dx} = 2(x+2)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+2)^2.2(x-3) - (x-3)^2.2(x+2)}{(x+2)^4}$$

$$= \frac{(x+2)^2.2(x-3) - (x-3)^2.2(x+2)}{(x+2)^4}$$

$$= \frac{2(x+2)(x-3)\{(x+2)-(x-3)\}}{(x+2)^4}$$

$$= \frac{2(x-3)\{x+2-x+3\}}{(x+2)^3}$$

$$= \frac{2(x-3)(5)}{(x+2)^3}$$

$$\underline{\frac{dy}{dx} = \frac{10(x-3)}{(x+2)^3}}$$

Example #2

$$\text{differentiate} \quad \frac{x}{\sqrt{(1+x^2)}}$$

$$u = x \qquad v = (1+x^2)^{1/2}$$

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}.2x(1+x^2)^{-1/2}$$

$$= x(1+x^2)^{-1/2}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(1+x^2)^{1/2}.1 - x.x(1+x^2)^{-1/2}}{(1+x^2)}$$

$$= \frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{(1+x^2)}$$

mult. top & bottom by $(1+x^2)^{1/2}$

$$= \frac{(1+x^2) - x^2}{(1+x^2)^{3/2}}$$

$$\underline{\frac{dy}{dx} = \frac{1}{(1+x^2)^{3/2}}}$$

Example #3

$$\text{differentiate} \quad \frac{1-x^2}{1+x^2}$$

$$u = 1 - x^2 \qquad v = 1 + x^2$$

$$y = \frac{u}{v}$$

$$\frac{du}{dx} = -2x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+x^2).(-2x) - (1-x^2).2x}{(1+x^2)^2}$$

$$= \frac{-(2x+2x^3) - (2x-2x^3)}{(1+x^2)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$$

$$\underline{\frac{dy}{dx} = -\frac{4x}{(1+x^2)^2}}$$

Parametric Equations

Parametric Equations

Both x and y are given as functions of another variable - called a **parameter** (eg 't'). Thus a pair of equations, called **parametric** equations, completely describe a single x-y function.

The d ifferentiation of functions given in parametric form is carried out using the Chain Rule.

Example #1

$$\text{find } \frac{dy}{dx} \text{ when } \quad x = t^2 \ , \quad y = 2t$$

$$x = t^2 \qquad\qquad y = 2t$$

$$\frac{dx}{dt} = 2t \qquad\qquad \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2 \cdot \frac{1}{2t}$$

$$\underline{\frac{dy}{dx} = \frac{1}{t}}$$

Example #2

given that $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$

show that $\dfrac{dy}{dx} = \tan 2\theta$

$$x = 3\cos\theta - \cos 3\theta \qquad y = 3\sin\theta - \sin 3\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta + 3\sin 3\theta\ , \quad \frac{dy}{d\theta} = 3\cos\theta - 3\cos 3\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{dx}{d\theta} = 3\cos\theta - 3\cos 3\theta.\frac{1}{-3\sin\theta + 3\sin 3\theta}$$

$$\frac{dy}{dx} = \frac{3\cos\theta - 3\cos 3\theta}{3\sin 3\theta - 3\sin\theta} = \frac{\cos\theta - \cos 3\theta}{\sin 3\theta - \sin\theta}$$

but $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

$\therefore \quad \cos(2\theta+\theta) - \cos(2\theta-\theta) = -2\sin 2\theta\sin\theta$

$\Rightarrow \quad \cos(3\theta) - \cos(\theta) = -2\sin 2\theta\sin\theta$

$\Rightarrow \quad \cos(\theta) - \cos(3\theta) = 2\sin 2\theta\sin\theta$

also $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$\therefore \quad \sin(2\theta+\theta) - \sin(2\theta-\theta) = 2\cos 2\theta\sin\theta$

$\Rightarrow \quad \sin(3\theta) - \sin(\theta) = 2\cos\theta\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{2\sin 2\theta\sin\theta}{2\cos 2\theta\sin\theta}$$

$$= \underline{\tan 2\theta}$$

Example #3

if $x = t^3 - t^2$ and $y = t^2 - t$,

find $\dfrac{dy}{dx}$ in terms of t

$$x = t^3 - t^2 \quad \therefore \quad \frac{dx}{dt} = 3t^2 - 2t$$

$$y = t^2 - t \quad \therefore \quad \frac{dy}{dt} = 2t - 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} . \frac{dt}{dx} \\ &= 2t - 1 . \frac{1}{3t^2 - 2} \\ &= \underline{\frac{2t - 1}{t(3t - 2)}} \end{aligned}$$

Implicit Equations

Explicit equations

Explicit equations are the type we are most familiar with eg y=f(x), y = $2x^2 + 3x - 5$ etc. where y is expressed in terms of x or some other variable.

Implicit equations

Implicit equations have the structure of being a mix of x and y terms eg $2x^2 + 3xy - 3y^2 = 5$, so y cannot be expressed in terms of x.

The method for solving equations of this type is to regard the whole expression as a function of x and to differentiate both sides of the equation. Any power of y is treated as a 'function of a function', as y is a function of x.

Example #1

find $\frac{dy}{dx}$ for the implicit function:

$$x^3 + 3y^4 - y^2 - 2x = 0$$

$$\frac{d(x^3)}{dx} + \frac{d(3y^4)}{dx} - \frac{d(y^2)}{dx} - \frac{d(2x)}{dx} = 0$$

$$3x^2 + 12y^3\frac{dy}{dx} - 2y\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx}(12y^3 - 2y) = 2 - 3x^2$$

$$\underline{\frac{dy}{dx} = \frac{2 - 3x^2}{12y^3 - 2y}}$$

Example #2

find $\frac{dy}{dx}$ for the implicit function:

$$\ln(y) = y\ln(x)$$

for $x > 0$ and $y > 0$

$$\ln(y) = y\ln(x)$$

$$\frac{d(\ln(y))}{dx} = \frac{d(y\ln(x))}{dx} \qquad (i$$

for the expression $\frac{d(y\ln(x))}{dx}$

let $u = y$ and $v = \ln(x)$

$$\frac{du}{dy} = \frac{du}{dx}.\frac{dy}{du} = \frac{dy}{dx} \qquad \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d(y\ln(x))}{dx} = y.\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

substituting into $(i$ for $\frac{d(y\ln(x))}{dx}$

$$\frac{1}{y}\frac{dy}{dx} = y.\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

rearranging

$$\frac{1}{y}\frac{dy}{dx} - \ln(x)\frac{dy}{dx} = \frac{y}{x}$$

multiplying each side by y

$$\frac{dy}{dx} - y\ln(x)\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx}(1 - y\ln(x)) = \frac{y^2}{x}$$

$$\underline{\frac{dy}{dx} = \frac{y^2}{x(1 - y\ln(x))}}$$

Example #3

find the gradient of the curve:

$x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4,1)$

$$\frac{d(x^2)}{dx} + \frac{d(2xy)}{dx} - \frac{d(2y^2)}{dx} + \frac{d(x)}{dx} = 2$$

$$2x + (2.y + 2x.\frac{dy}{dx}) - 4y.\frac{dy}{dx} + 1 = 0$$

$$\therefore \frac{dy}{dx}(2x - 4y) = -1 - 2x - 2y$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 4y}$$

when $x = -4$ and $y = 1$

$$\frac{dy}{dx} = \frac{-1 - 2(-4) - 2(1)}{2(-4) - 4(1)} = \frac{-1 + 8 - 2}{-8 - 4}$$

$$= \frac{5}{-12} = -\frac{5}{12}$$

gradient at (-4,1) is $-\frac{5}{12}$

Differential Equations

Definition

An equation containing any **differential coefficients** is called a differential equation.

$$\text{differential coefficients: } \frac{dy}{dx}\ ,\ \frac{d^2y}{dx^2}\ ,\ \frac{d^3y}{dx^3}\ \ldots$$

The solution of a differential equation is an equation relating x and y and containing **no** differential coefficients.

General & Particular Solution

The **General Solution** includes some unknown constant in the solution of a differential equation.

When some data is given, say the coordinates of a point, then a **Particular Solution** can be formed.

example

differential equation: $\frac{dy}{dx} = 4$

general solution: $y = 4x + c$

(where c is an unknown constant)

if we are given that $x = 3$ when $y = 5$

then $5 = 12 + c$, so $c = -7$

particular solution: $y = 4x - 7$

Example #1

$$\text{find } \frac{d^3y}{dx^3} \text{ when } y = \frac{3}{x}$$

$$y = \frac{3}{x} = 3x^{-1}$$

$$\therefore \frac{dy}{dx} = 3(-1)x^{-2} = -\frac{3}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} = -3(-2)x^{-3} = \frac{6}{x^3}$$

$$\therefore \frac{d^3y}{dx^3} = 6(-3)x^{-4} = \underline{-\frac{18}{x^4}}$$

Example #2

$$\text{given that } \quad y = Ax^2 + B\ln x + C$$

$$\text{show that } \frac{d^2y}{dx^2} = \frac{1}{x}.\frac{dy}{dx}$$

$$y = Ax^2 + B\ln x + C$$

$$\frac{dy}{dx} = 2Ax + \frac{B}{x}$$

$$\frac{d^2y}{dx^2} = 2A - \frac{B}{x^2}$$

$$= \frac{1}{x}\left(2Ax - \frac{B}{x}\right)$$

$$\underline{\frac{d^2y}{dx^2} = \frac{1}{x}.\frac{dy}{dx}}$$

Points of Inflection(Inflexion)

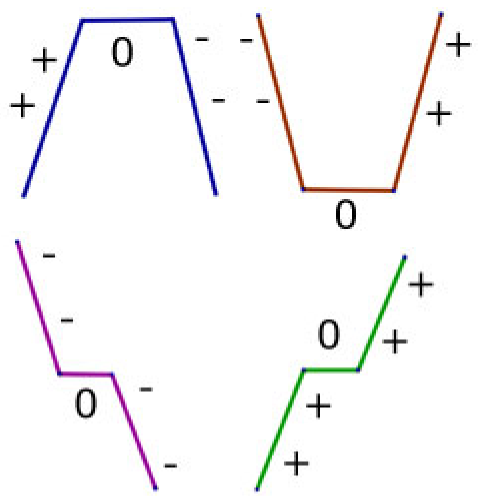
The value of the second derivative can give an indication whether at a point a function has a maximum, minimum or an inflection.These are all called **stationary points**.

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad \text{maximum(-ve)}$$

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad \text{minimum(+ve)}$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \text{inflection(zero)}$$

A point of inflection has a zero gradient, but the point is not a maximum or a minimum value.



It is where the gradient of a curve decreases(or increases)to zero before increasing(or decreasing)again, but not changing from a negative to a positive value or vice versa.

Example

Find the stationary points of the function:

$y = 3x^4 - 4x^3 - 12x^2 + 5$

$$y = 3x^4 - 4x^3 - 12x^2 + 5$$

$$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x+1)(x-2)$$

$$\therefore \ x \text{ has roots } \ 0,\text{-}1,2$$

$$\frac{d^2y}{dx^2} = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

$$\text{when } \ x = 0 \quad \frac{d^2y}{dx^2} = -24 < 0 \quad \therefore \text{max.}(0,10)$$

$$\text{when } \ x = -1 \quad \frac{d^2y}{dx^2} = 36 > 0 \quad \therefore \text{min.}(-1,5)$$

$$\text{when } \ x = 2 \quad \frac{d^2y}{dx^2} = 72 > 0 \quad \therefore \text{min.}(2,22)$$

Notes